PLANCKS 2023 Problems Booklet

Dear contestants Welcome to PLANCKS 2023

- The language used in the competition is English.
- The contest consists of **10 exercises**, **each worth 10 points**. Subdivisions of points are indicated in the exercises.
- All exercises have to be handed in on a separate sheet of paper. Please mark each sheet with your team name and the exercise number. **Marking each sheet is necessary for an exercise to be scored**.
- When a problem is unclear, a participant can ask, via the crew, for clarification. If the response is relevant to all teams, the jury will provide this information to the other teams.
- You are allowed to use a dictionary: English to your native language.
- You are allowed to use a non-programmable, not-graph calculator (But scientific is okay).
- No books or other sources, except for this exercise booklet and a dictionary, are to be consulted during the competition.
- The jury has the right to disqualify teams for misbehavior or breaking the rules.
- The use of hardware (including phones, tablets etc.) is not approved, except for watches and medical equipment. Please leave your phones in an envelope.
- In situations to which no rule applies, the OC decides.

May the best physics team win!

Milan, 14/05/2023

1 Quantum Mechanics

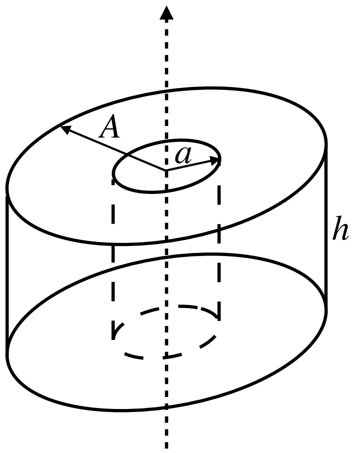


Figure 1: A cavity between two coaxial cylinders.

Find the energetic levels of a particle living in a cavity between two coaxial cylinders of height h and radii a, A ($a < A$), as depicted in Fig. 1. Consider the following cases:

i) **(1 point)** $h \gg A$ (both cylinders extend indefinitely along their axis);

ii) **(3 points)** The cavity is closed by two lids on the top and bottom ends;

iii) **(6 points)** A uniform magnetic field $\vec{B}$, directed along the cylinder axis, is present only inside the smallest cylinder, but is zero anywhere else (including in the cavity).

For the radial part, it is enough to obtain the equation(s) whose solutions give the energy levels.

Hints:

i) The gradient, curl and Laplacian in cylindrical coordinates are

$$\vec{\nabla} = \frac{\partial}{\partial z}\hat{z} + \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi}\,, \tag{1}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\hat{z} + \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi}\,, \tag{2}$$

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\,. \tag{3}$$

ii) The Bessel functions are the solutions of the differential equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0 \quad \alpha > 0\,. \tag{4}$$

The Bessel functions of the first kind, $y = J_\alpha(x)$, are regular at $x \to 0$, while those of the second kind, $y = Y_\alpha(x)$, diverge in the same limit. In Fig. 2 some of the Bessel functions are plotted.

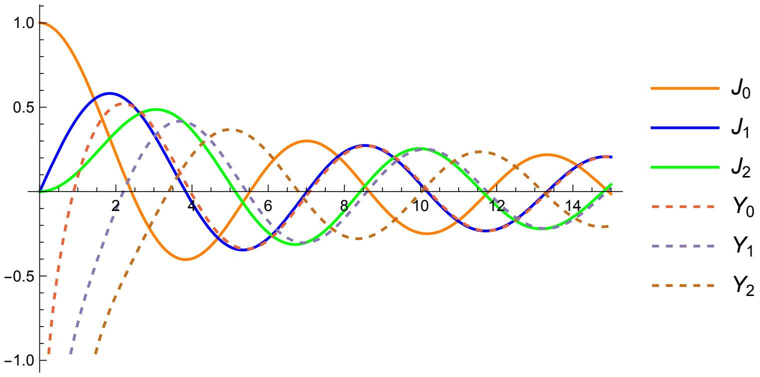


Figure 2: Bessel functions of the first and second kind.

Solutions

(i) and (ii) In the region between the two cylinders, the Hamiltonian reduces to the free-particle case for i) and ii) (we will discuss the boundary conditions in due time). The Hamiltonian, inside the cavity

$$H = -\hbar^2 \frac{\nabla^2}{2m} \,. \tag{5}$$

The first step, is use write the Laplacian operator in cylindrical coordinates (z, r, ϕ), using hint i). We note that z is completely separated. Thus, in order to find the eigenfunctions of the Hamiltonian, we assume for them a factorised form

$$\Psi^E(z, r, \phi) = \zeta^{E_z}(z)\, \psi^{E_{r\phi}}(r, \phi) \,, \tag{6}$$

with the corresponding eigenvalue

$$E = E_z + E_{r\phi} \,. \tag{7}$$

Cases i) and ii) differ only in the z part:

- for case i), we have a free-particle problem, where eigenfunctions are plane waves

$$\zeta^{E_z}(z) = \zeta^k(z) = N_{k,z}\, e^{ikx} \,, \tag{8}$$

  and

$$E_z = E_z^k = \frac{\hbar^2 k^2}{2m} \,; \tag{9}$$

- for case ii), we instead have a one-dimensional infinite-well problem, with the boundary conditions $\zeta^{E_z}(0) = \zeta^{E_z}(h) = 0$. The solutions are

$$\zeta^{E_z}(z) = \zeta^{n_z}(z) = N_{n_z,z} \sin \frac{2\pi n_z x}{h} \,, \tag{10}$$

  and the energy eigenvalues

$$E_z = E_z^{n_z} = \frac{\hbar^2 n_z^2 \pi^2}{2mh^2} \,. \tag{11}$$

Let us now turn to the less trivial r, ϕ part. In this case, the equation for the eigenstates of the Hamiltonian reads

$$-\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right] \psi^{E_{r\phi}}(r, \phi) - E_{r\phi} \psi^{E_{r\phi}}(r, \phi) = 0 \,. \tag{12}$$

The part involving derivatives w.r.t. ϕ can be written in term of the (only component of the) angular momentum L:

$$L = -i\hbar \frac{\partial^2}{\partial \phi^2} \tag{13}$$

Rotational invariance allows us to replace the operator L with its eigenvalue $\hbar l$. We rewrite Eq. 12 as

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) - \frac{1}{r^2} l^2 + \frac{2mE_{rl}}{\hbar^2} \right] \rho^{E_{rl}}(r) = 0 \,, \tag{14}$$

where, after diagonalising the angular momentum, we have factored out the ϕ-dependent part of the eigenfunction as

$$\psi^{E_{rl}}(r,\phi) = \frac{1}{\sqrt{2\pi}} e^{il\phi} \rho^{E_{rl}}(r)\,, \tag{15}$$

and we have re-labeled the energy eigenvalue exposing its dependence on l.

For the radial part, we multiply both sides of Eq. 14 by r^2 and let the derivatives act, thus obtaining

$$\left(r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{2mE_{rl}}{\hbar^2} r^2 - l^2 \right) \rho^{E_{rl}}(r) = 0\,. \tag{16}$$

We almost recognize the differential equation satisfied by the Bessel functions (hint ii)), which we can obtain by introducing the variable

$$u(r) = \frac{\sqrt{2mE_{rl}}}{\hbar} r\,. \tag{17}$$

We have indeed

$$\left(u^2 \frac{\partial^2}{\partial u^2} + u\frac{\partial}{\partial u} + u^2 - l^2 \right) \rho^{E_{rl}}(u) = 0\,. \tag{18}$$

Therefore, we can obtain the solutions, which we can write as linear combinations of the Bessel functions of first and second kind:

$$\rho^{E_{rl}}(u) = N^J_{l,z} J_l(u) + N^Y_{l,z} Y_l(u)\,. \tag{19}$$

Note that, since the cavity does not extend in the region $r \to 0$, we must keep both kind of Bessel functions. In order to find the allowed energy levels, we must impose the boundary conditions on the radial part. Wavefunctions are required to vanish at th edges of the cavity, thus one must have

$$\rho^{E_{rl}}(r=a) = \rho^{E_{rl}}(r=A) = 0\,. \tag{20}$$

We therefore have the following two relations

$$N^J_{l,z} J_l(u(a)) + N^Y_{l,z} Y_l(u(a)) = 0\,, \tag{21}$$
$$N^J_{l,z} J_l(u(A)) + N^Y_{l,z} Y_l(u(A)) = 0\,, \tag{22}$$

leading to the (normalisation-constant independent) equation

$$\frac{J_l(u(a))}{J_l(u(A))} = \frac{Y_l(u(a))}{Y_l(u(A))}\,, \tag{23}$$

whose solutions are the allowed energy values.

(iii) The subtle part here is the role of the magnetic field which, despite being non-zero only outside the cavity, it affects the energy eigenfunctions. The Hamiltonian in the presence of a magnetic field can be obtained from the standard one by applying the minimal-coupling prescription, i.e. by replacing

$$\vec{p} \to \vec{p} - \frac{e\vec{A}}{c}\,, \tag{24}$$

$\vec{A}$ being the electromagnetic vector potential. We must thus derive the vector potential, which is related to the magnetic field as

$$\vec{B} = \vec{\nabla} \times \vec{A} . \tag{25}$$

In our case, $\vec{B} = B_z \hat{z}$ if $r < a$, else $\vec{B} = 0$.
The curl operator, expressed in cylindrical coordinates, given by hint i), reads

$$\vec{\nabla} \times \vec{A} = \frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\hat{z} + \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi} \tag{26}$$

The symmetries of the problem (ϕ-rotational and z-traslational invariance), imply $\partial_z A_i = \partial_\phi A_i = 0$. Eq. 25 thus become:

$$\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} = B_z \qquad \text{for } r < a\,, \tag{27}$$

$$\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} = 0 \qquad \text{for } r > a\,, \tag{28}$$

$$\frac{\partial A_z}{\partial r} = 0\,. \tag{29}$$

The solutions are

$$A_\phi = \frac{B_z r}{2} + \frac{c}{r} \qquad \text{for } r < a\,, \tag{30}$$

$$A_\phi = \frac{c'}{r} \qquad \text{for } r > a\,, \tag{31}$$

$$A_z = c''\,. \tag{32}$$

If we require that A_ϕ is regular at $r = 0$ then we must set $c = 0$; continuity at $r = a$ implies $c' = \frac{B_z a^2}{2r}$. Thus, neglecting other constant terms which give only phase-shifts to the wavefunctions, the vector potential inside the cavity is

$$\vec{A} = \frac{B_z a^2}{2r}\hat{\phi}\,. \tag{33}$$

(The same results for A_ϕ can be obtained with Stoke's theorem.)
We now apply the minimal-coupling prescription Eq. 24. Only the ϕ component changes, as

$$p_\phi = \frac{1}{r}\left(-i\hbar\frac{\partial}{\partial\phi} - \frac{B_z a^2 e}{2c}\right). \tag{34}$$

Thus, the ϕ dependent part of the wavefunction must now be an eigenfunction of the operator

$$\tilde{L} = -i\hbar\frac{\partial}{\partial\phi} - \frac{B_z a^2 e}{2c}\,, \tag{35}$$

i.e. it must satisfy the differential equation

$$\left(-i\hbar\frac{\partial}{\partial\phi} - \frac{B_z a^2 e}{2c}\right)\varphi(\phi) = \hbar\tilde{l}\varphi(\phi)\,, \tag{36}$$

with the periodicity condition
$$\varphi(0) = \varphi(2\pi)\,. \tag{37}$$
The soluton is trivial
$$\varphi(\phi) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{i}{\hbar}\phi\left(\hbar\tilde{l} + \frac{B_z a^2 e}{2c}\right)\right], \tag{38}$$
but the periodicity condition now requires that
$$\tilde{l} + \frac{B_z a^2 e}{2\hbar c} \in \mathbb{N}\,. \tag{39}$$
The energy levels are affected because the radial part of the wavefunction must now satisfy
$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\tilde{l}^2 + \frac{2mE_{r\tilde{l}}}{\hbar^2}\right]\rho^{E_{r\tilde{l}}}(r) = 0\,, \tag{40}$$
in which, at variance with Eq. 14, $\tilde{l}$ is not an integer anymore, because of the term proportional to the magnetic field.

A comment is worth here: what we have derived with sweat and pain should somehow surprise us. Even if there is no magnetic field in the cavity, i.e. the particle does not experience any force, still the presence of the magnetic field inside the inner region (not accessible by the particle) has an effect on the energy levels. Otherwise said, contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where all the fields (and therefore the forces on the particles) vanish. This is the *Aharonov-Bohm* effect (Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in the quantum theory," Phys. Rev. **115** (1959), 485-491), which has been measured experimentally.

2 Thermodynamics

Let's consider a spherical soap bubble containing n moles of a perfect gas, whose conductive surface has a charge q. The bubble is at atmospheric conditions: let T_0 be the temperature and p_0 the pressure, while γ_0 is the constant of surface tension of the **uncharged** water and soap layer.

i) **(2 points)** Show that the constant of surface tension depends on the charge;

ii) **(4 points)** Determine the equilibrium conditions for the charged bubble. Verify that there is always one and only one value that satisfies these conditions;

iii) **(4 points)** Study the stability of the equilibrium.

Solutions

(i) We can compute the differential of the free energy for the membrane of the bubble:

$$\mathrm{d}F_b = -s_b \mathrm{d}T + \gamma \mathrm{d}\Sigma + \frac{q}{C}\mathrm{d}q \ ,$$

where $C = 4\pi\epsilon_0 r$ is the electric capacitance of the bubble, r is its radius and Σ its surface density, which, since there are two interfaces, can be written as $\Sigma = 8\pi r^2$. We can obtain from Maxwell relations:

$$\left.\frac{\partial\gamma}{\partial q}\right|_{\Sigma,T} = -\left.\frac{\partial(q/C)}{\partial\Sigma}\right|_{q,T} \ ,$$

which implies

$$\left.\frac{\partial\gamma}{\partial q}\right|_{T} = -\frac{q}{64\pi^2\epsilon_0 r^3}.$$

By integrating this equation we get

$$\gamma(T,r,q) = \gamma_0(T) - \frac{q^2}{128\pi^2\epsilon_0 r^3}.$$

(ii) At equilibrium, the "bubble" system (gas and film) is at the thermostat temperature T_0. To find the equilibrium conditions, the minimum of the potential $\tilde{G}_0$ is sought, with respect to the unique internal variable V (or the radius r).

$$\tilde{G}_0 = F(T_0, V, q) + p_0 V \ .$$

Let us call p_g the pressure of the gas inside the bubble. At constant q and T $\mathrm{d}F$ reads:

$$\mathrm{d}F = -p_g \mathrm{d}V + \gamma \mathrm{d}\Sigma \ .$$

Hence

$$\left.\frac{\partial\tilde{G}_0}{\partial r}\right|_{T,q} = 4\pi r^2\left[(p_0 - p_g) + \frac{4r_0}{r} - \frac{q^2}{32\pi^2\epsilon_0 r^4}\right] \ .$$

Thus, we can obtain the equilibrium condition:

$$p_g - p_0 = \frac{4\gamma_0(T_0)}{r_e} - \frac{q^2}{32\pi^2\epsilon_0 r_e^4} = \frac{4\gamma_0(T_0)}{r_e} - \frac{\sigma^2}{2\epsilon_0} \ ,$$

where $\sigma = \frac{q}{4\pi r^2}$ is the surface charge density. It is possible to notice that the term $\frac{\sigma^2}{2\epsilon_0}$ is equal to the electrostatic pressure p_{el}, so we can write

$$p_g - p_0 = \frac{4\gamma_0(T_0)}{r_e} - p_{\mathrm{el}} \ .$$

The right hand side of this equation could be negative *a priori*, which would mean that the pressure inside the bubble is higher than the pressure outside. However, this would not be an acceptable result, since we assumed the bubble to be spherical, and thus with

$$\frac{4\gamma_0(T_0)}{r_e} - p_{\mathrm{el}} = 4\gamma > 0 \ .$$

After having imposed $\gamma > 0$, we can obtain the equilibrium condition:

$$\frac{3nRT_0}{4\pi} = p_0 r_e^3 + 4\gamma_0 r_e^2 - \frac{q^2}{32\pi^2\epsilon_0 r_e} \ ,$$

where p_e was obtained from the ideal gas law. The right hand side of this equation is a monotonically increasing function (for $r_e > 0$), which tends to $-\infty$ if $r_e \to 0$ and to $+\infty$ if $r_e \to +\infty$. Hence, it allows one unique solution.

(iii) In order to study the stability of the equilibrium we need to determine the sign of the second derivative of $\tilde{G}_0$:

$$\left.\frac{\partial^2 \tilde{G}_0}{\partial r^2}\right|_{T_0}^{\text{eq.}} = 4\pi r_e^2 \left[-\left.\frac{\partial p_g}{\partial V}\right|_{T_0} \frac{\mathrm{d}V}{\mathrm{d}r} - \frac{4\gamma_0}{r_e^2} + \frac{q^2}{8\pi^2\epsilon_0 r_e^5} \right]^{\text{eq.}} \ .$$

Since q appears only in a positive term, it will not threaten the stability of the bubble. Thus, as soon as the bubble is spherical:

$$\left.\frac{\partial^2 \tilde{G}_0}{\partial r^2}\right|_{T_0}^{\text{eq.}} > 0 \ ,$$

which means the equilibrium is stable.

3 Quantum Optics

Rabi oscillations — A traveling two-level atom interacts with a classical single-mode oscillating field inside a lossless microwave cavity. The detuning between the atomic transition and the cavity field is $\Delta\omega$ and the Hamiltonian describing the atom-cavity field system in the interaction picture is given by:

$$H_{\text{int}} = \hbar\frac{\Delta\omega}{2}\sigma_3 - \hbar\frac{\Omega_0}{2}\sigma_1\,, \tag{41}$$

where $\sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|$ and $\sigma_1 = |e\rangle\langle g| + |g\rangle\langle e|$ are Pauli operators, $|e\rangle$ and $|g\rangle$ refer to the excited and ground state of the atom, respectively, and Ω_0 is the Rabi frequency.

Assuming that the atom is initially in the ground state, namely, $|\psi_0\rangle = |g\rangle$:

i) **(2 points)** find the evolved state:

$$|\psi_t\rangle = \exp\left(-i\frac{H_{\text{int}}\,t}{\hbar}\right)|\psi_0\rangle. \tag{42}$$

Then, write the analytic expression of the probability of finding the atom in the excited state $P_e(t) = |\langle e|\psi_t\rangle|^2$ and comment the result in the following cases:

ii) **(1 point)** when $\Delta\omega = 0$ (resonance);

iii) **(1 point)** when $\Delta\omega \gg \Omega_0$ (large detuning).

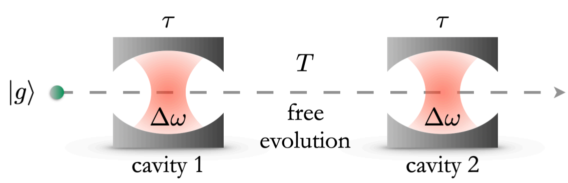
If the interaction time is set to $t = \pi/\Omega_0$ (corresponding to a π-pulse) and $\Delta\omega \neq 0$:

iv) **(1 point)** write $P_e(t = \pi/\Omega_0)$ as a function of the quantity $x = \Delta\omega/\Omega_0$ and find the value $x > 0$ at which the first minimum occurs.

Ramsey fringes — Now we consider a setup involving two cavities, as displayed in the figure.

The interaction between the atom and the field inside each cavity is still described by the Hamiltonian (41), whereas, during the free evolution between the two cavities, the Hamiltonian reads (in the interaction picture):

$$H_{\text{free}} = \hbar\frac{\Delta\omega}{2}\sigma_3. \tag{43}$$



We also assume that the two cavity fields have the same phase that we set equal to zero. The interaction time inside both the cavities is set to $\tau = \pi/(2\,\Omega_0)$, corresponding to a $\frac{\pi}{2}$–pulse, and the free evolution lasts for a time $T = \xi\tau$, with $\xi \in \mathbb{N}$.

If the initial state of the atom is still $|\psi_0\rangle = |g\rangle$ and the evolved state after the whole evolution (cavity 1 — free evolution — cavity 2) is $|\psi_{\text{out}}\rangle$:

v) **(2 points)** prove that the probability of finding the atom in the excited state $\tilde{P}_e = |\langle e|\psi_{\text{out}}\rangle|^2$ as a function of $x = \Delta\omega/\Omega_0$ can be written as:

$$\tilde{P}_e(x;\xi) = g(x)\, f_\xi(x), \tag{44}$$

where:

$$g(x) = \frac{4}{1+x^2}\, \sin^2\left(\frac{\pi}{4}\sqrt{1+x^2}\right),$$

and

$$f_\xi(x) = \left[\cos\left(\frac{\pi}{4}\xi x\right)\cos\left(\frac{\pi}{4}\sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}}\sin\left(\frac{\pi}{4}\xi x\right)\sin\left(\frac{\pi}{4}\sqrt{1+x^2}\right)\right]^2,$$

vi) **(3 points)** show that the smallest value $x_{\text{m}} > 0$ such that $\tilde{P}_e(x_{\text{m}};\xi) = 0$ scales as $1/\xi$ in the limit $\xi \gg 1$.

Solutions

— Rabi oscillations

(i) It is useful, but not necessary to solve the problem, rewriting the Hamiltonian (41) in the following compact form:

$$H_{\text{int}} = \hbar\frac{\Omega}{2}\vec{n}\otimes\vec{\sigma}$$

where $\Omega = \sqrt{\Omega_0^2 + (\Delta\omega)^2}$, we introduced the unit vector:

$$\vec{n} = \left(-\frac{\Omega_0}{\Omega}, 0, \frac{\Delta\omega}{\Omega}\right),$$

and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of the Pauli operastors. Since $(\vec{n}\otimes\vec{\sigma})^2 = \mathbb{I}$, it easy to show that:

$$U_{\text{int}}(t) = \exp\left(-i\frac{H_{\text{int}}\, t}{\hbar}\right) = \cos\left(\frac{\Omega t}{2}\right) - i\sin\left(\frac{\Omega t}{2}\right)\vec{n}\otimes\vec{\sigma}.$$

If we associate the two vectors:

$$|e\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad |g\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with the atom states $\{|e\rangle, |g\rangle\}$, respectively, the evolution operator $U_{\text{int}}(t)$ can be written in the matrix from:

$$U_{\text{int}}(t) = \begin{pmatrix} A(\Omega_0, \Delta\omega, t) & B(\Omega_0, t), \\ B(\Omega_0, t) & C(\Omega_0, \Delta\omega, t) \end{pmatrix}, \tag{45}$$

where

$$A(\Omega_0, \Delta\omega, t) = \cos\left(\frac{\Omega t}{2}\right) - i\frac{\Delta\omega}{\Omega}\sin\left(\frac{\Omega t}{2}\right),$$

$$B(\Omega_0, t) = i\frac{\Omega_0}{\Omega}\sin\left(\frac{\Omega t}{2}\right),$$

$$C(\Omega_0, \Delta\omega, t) = \cos\left(\frac{\Omega t}{2}\right) + i\frac{\Delta\omega}{\Omega}\sin\left(\frac{\Omega t}{2}\right) \equiv A^*(\Omega_0, \Delta\omega, t).$$

Thereafter, considering the initial state $|\psi_0\rangle = |g\rangle$, we find:

$$\begin{aligned} |\psi_t\rangle &= U_{\text{int}}(t)|\psi_0\rangle, \\ &= C(\Omega_0, \Delta\omega, t)\,|g\rangle + B(\Omega_0, t)\,|e\rangle. \end{aligned}$$

The requested probability $P_e(t)$ is thus given by:

$$P_e(t) = |\langle e|\psi_t\rangle|^2 = \left(\frac{\Omega_0}{\Omega}\right)^2 \sin^2\left(\frac{\Omega t}{2}\right). \tag{46}$$

(ii) At resonance, $\Delta\omega = 0$ and $\Omega = \Omega_0$, and we find:

$$P_e(t) = \sin^2\left(\frac{\Omega_0 t}{2}\right) = \frac{1}{2}\left[1 - \cos\left(\Omega_0 t\right)\right],$$

and the atom periodically oscillates between its ground and exited state (Rabi oscillations).

(iii) In the presence of large detuning, $\Delta\omega \gg \Omega_0$ and $\Omega \approx \Delta\omega$, and we obtain:

$$P_e(t) = \left(\frac{\Omega_0}{\Delta\omega}\right)^2 \sin^2\left(\frac{\Delta\omega t}{2}\right) \approx 0,$$

the atom remains in its initial state $|\psi_0\rangle = |g\rangle$ (up to a global phase).

(iv) We now set $t = \pi/\Omega_0$ and Eq. (46) becomes:

$$P_e(t = \pi/\Omega_0) = \frac{\Omega_0^2}{\Omega_0^2 + (\Delta\omega)^2} \sin^2\left[\frac{\pi}{2\,\Omega_0}\sqrt{\Omega_0^2 + (\Delta\omega)^2}\right],$$

that, introducing the parameter $x = \Delta\omega/\Omega_0$, rewrites as:

$$P_e(x) = \frac{1}{1+x^2} \sin^2\left(\frac{\pi}{2}\sqrt{1+x^2}\right).$$

The minima of $P_e(x)$ occur when the probability vanishes, namely, when:

$$\frac{\pi}{2}\sqrt{1+x^2} = k\pi, \quad k \in \mathbb{N}$$

and, for $k = 1$, corresponding to the first minimum, we have:

$$x = \sqrt{3}.$$

— Ramsey fringes

To solve the second part of the problem, we should find the evolved state $|\psi_{\tau,T}\rangle$, that is formally given by:

$$|\psi_{\tau,T}\rangle = \underbrace{U_{\text{int}}(\tau)}_{\text{cavity 2}}\, \underbrace{U_{\text{free}}(T)}_{\text{free evol.}}\, \underbrace{U_{\text{int}}(\tau)}_{\text{cavity 1}}\, |\psi_0\rangle,$$

where

$$\begin{aligned} U_{\text{free}}(T) &= \exp\left(-i\frac{H_{\text{free}}\,T}{\hbar}\right), \\ &= \begin{pmatrix} \exp(-i\Delta\omega T/2) & 0 \\ 0 & \exp(i\Delta\omega T/2) \end{pmatrix}, \end{aligned}$$

is the evolution operator associated with the free evolution between the cavities. After some calculations we obtain:

$$U_{\text{int}}(\tau)U_{\text{free}}(T)U_{\text{int}}(\tau) = \begin{pmatrix} A_\tau^2\, e^{-i\Delta\omega T/2} + B_\tau^2\, e^{i\Delta\omega T/2} & A_\tau\, B_\tau\, e^{-i\Delta\omega T/2} + B_\tau C_\tau\, e^{i\Delta\omega T/2} \\ A_\tau\, B_\tau\, e^{-i\Delta\omega T/2} + B_\tau\, C_\tau\, e^{i\Delta\omega T/2} & B_\tau^2\, e^{-i\Delta\omega T/2} + C_\tau^2\, e^{i\Delta\omega T/2} \end{pmatrix},$$

where $A_\tau = A(\Omega_0, \Delta\omega, \tau)$, $B_\tau = B(\Omega_0, \tau)$ and $C_\tau = C(\Omega_0, \Delta\omega, \tau)$ have been introduced in Eq. (45).

(v) It is now easy to show that, in general:

$$\tilde{P}_e(\tau,T) = |\langle e|\psi_{\tau,T}\rangle|^2$$
$$= 4\left(\frac{\Omega_0}{\Omega}\right)^2 \sin^2\left(\frac{\Omega\tau}{2}\right)\left[\cos\left(\frac{\Delta\omega T}{2}\right)\cos\left(\frac{\Omega\tau}{2}\right) - \frac{\Delta\omega}{\Omega}\sin\left(\frac{\Delta\omega T}{2}\right)\sin\left(\frac{\Omega\tau}{2}\right)\right]^2. \tag{47}$$

If we put $\tau = \pi/(2\,\Omega_0)$, $T = \xi\tau = \xi\pi/(2\,\Omega_0)$ and we introduce again $x = \Delta\omega/\Omega_0$, we arrive at:

$$\tilde{P}_e(x;\xi) = \frac{4\,\sin^2\left(\frac{\pi}{4}\sqrt{1+x^2}\right)}{1+x^2}\left[\cos\left(\frac{\pi}{4}\xi x\right)\cos\left(\frac{\pi}{4}\sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}}\sin\left(\frac{\pi}{4}\xi x\right)\sin\left(\frac{\pi}{4}\sqrt{1+x^2}\right)\right]^2 \tag{48}$$

$$= g(x)\, f_\xi(x), \tag{49}$$

with:

$$g(x) = \frac{4\,\sin^2\left(\frac{\pi}{4}\sqrt{1+x^2}\right)}{1+x^2}$$

and

$$f_\xi(x) = \left[\cos\left(\frac{\pi}{4}\xi x\right)\cos\left(\frac{\pi}{4}\sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}}\sin\left(\frac{\pi}{4}\xi x\right)\sin\left(\frac{\pi}{4}\sqrt{1+x^2}\right)\right]^2$$

Note that $\tilde{P}_e(x;0) = P_e(x)$, as we may expect.

(vi) We note that the "envelope" $g(x)$ of Eq. (49) vanishes if:

$$\frac{\pi}{4}\sqrt{1+x^2} = k\pi, \quad k \in \mathbb{N}$$

thus it becomes zero for the first time at $x = \sqrt{15}$, therefore, to find the first value $x_\mathrm{m} > 0$ giving $\tilde{P}_e(x_\mathrm{m}) = 0$, we can focus on the interval $0 \le x \le \sqrt{15}$ and investigate the behaviour of $f_\xi(x)$.

The function $f_\xi(x)$ vanishes when:

$$\cos\left(\frac{\pi}{4}\xi x\right)\cos\left(\frac{\pi}{4}\sqrt{1+x^2}\right) = \frac{x}{\sqrt{1+x^2}}\sin\left(\frac{\pi}{4}\xi x\right)\sin\left(\frac{\pi}{4}\sqrt{1+x^2}\right),$$

or, more simply, when:

$$\underbrace{\tan\left(\frac{\pi}{4}\xi x\right)}_{G_\xi(x)} = \underbrace{\frac{\sqrt{1+x^2}}{x}\cot\left(\frac{\pi}{4}\sqrt{1+x^2}\right)}_{F(x)}. \tag{50}$$

To estimate the value of $x_\mathrm{m} > 0$ we are looking for, we can proceed as follows. The asymptotes of the function $G_\xi(x)$ of Eq. (50) occur at:

$$\frac{\pi}{4}\xi|x| = k\frac{\pi}{2} \quad \to \quad |x| = k\frac{2}{\xi}$$

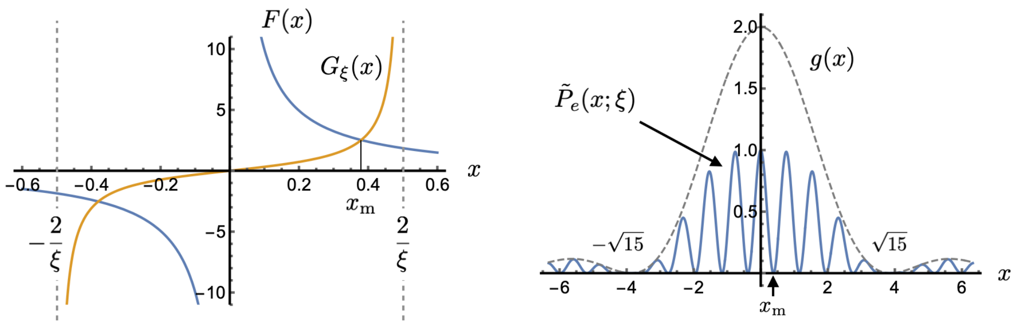


Figure 3: Left: Plot of the functions $G_\xi(x)$ and $F(x)$ for $x \in (-2/\xi, 2/\xi)$ with $\xi = 4$. Note the first two asymptotes at $x = \pm 2/\xi$ and the intersection at $x_\mathrm{m} > 0$. Right: Plot of $\tilde{P}_e(x;\xi)$ as a function of x for $\xi = 4$ (solid line). We also plot the "envelope" $g(x)$ with its first two minima at $x = \pm\sqrt{15}$ (dashed line).

with $k \in \mathrm{N}$, therefore, the first asymptote will be at $|x| = 2/\xi$. As a matter of fact, for $0 \le x \le 2/\xi$ we have:

$$G_\xi(x) \ge 0,$$

with $G_\xi(x) = 0$. Now we focus on the r.h.s. of Eq. (50) still in the interval $0 \le x \le 2/\xi$, where:

$$F(x) = \underbrace{\frac{\sqrt{1+x^2}}{x}}_{F_1(x)>0} \; \underbrace{\cot\left(\frac{\pi}{4}\sqrt{1+x^2}\right)}_{F_2(x)>0}$$

is clearly positive and, being:

$$\begin{aligned} F_1'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{1+x^2}}{x}\right) \\ &= -\frac{1}{x^2\sqrt{1+x^2}} < 0, \end{aligned}$$

and:

$$\begin{aligned} F_2'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\cot\left(\frac{\pi}{4}\sqrt{1+x^2}\right)\right] \\ &= -\frac{\pi x}{4\sqrt{1+x^2}\,\sin^2\left(\frac{\pi}{4}\sqrt{1+x^2}\right)} \le 0, \end{aligned}$$

we conclude that:

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = F_1'(x)\,F_2(x) + F_1(x)\,F_2'(x) < 0,$$

that is: $F(x)$ is a monotonic decreasing function in the interval $0 \le x \le 2/\xi$.

Eventually, we can state that Eq. (50) has a solution $0 \le x_\mathrm{m} \le 2/\xi$ leading to $\tilde{P}_e(x_\mathrm{m}) = 0$, and, in the limit $\xi \gg 1$, we obtain $x_\mathrm{m} \approx 2/\xi$.

As an example, Figure 3 shows the plots of $G_\xi(x)$, $F(x)$ (left panel) with the solution x_m and of the probability $\tilde{P}_e(x_\mathrm{m};\xi) = 0$ (right panel) for $\xi = 4$.

4 Structure of matter

Counting Spin-polarized Electrons - Consider an electron gas at temperature $T = 0$. Initially neglect the Coulombic electron-electron interaction. Assume that this electron gas is immersed in a static uniform magnetic field $B = 8.64$ T coupled to the electron spins (i.e. neglect the field effect on the orbital motion of the electrons). Let the length quantity L have the following value: $L = 41.1$ nm.

i) **(0.5 points)** Derive an expression for the maximum number density $n = N/V$ such that in the ground state the electron-spin magnetic moments of all N electrons align to the magnetic field, and evaluate the value of this density (in m^{-3}).

ii) **(1 point)** Describe the electrons of a metallic nanoparticle adopting a radically simplified model: a cubic volume $V = L \times L \times L$. Evaluate $N_{\max} = n\,V$ using the result of the previous question.

iii) **(1 point)** Apply periodic boundary conditions (PBC) to the free-electron wave functions in the $L \times L \times L$ box. By taking the resulting discrete single-electron energy levels and degeneracies into account, what is the actual maximum number $N_{\max}^{\mathrm{PBC}}$ that are 100% spin-aligned in the same magnetic field?

iv) **(2 points)** Modify the nanoparticle model to a cubic-shaped $L \times L \times L$ infinitely-deep potential well, thus replacing the PBC of question iii) with null boundary conditions (NBC) to the electron wave functions. By taking the resulting discrete single-electron energy levels and degeneracies into account, what maximum number $N_{\max}^{\mathrm{NBC}}$ of electrons with 100% spin alignment (in the same magnetic field) does this model yield?

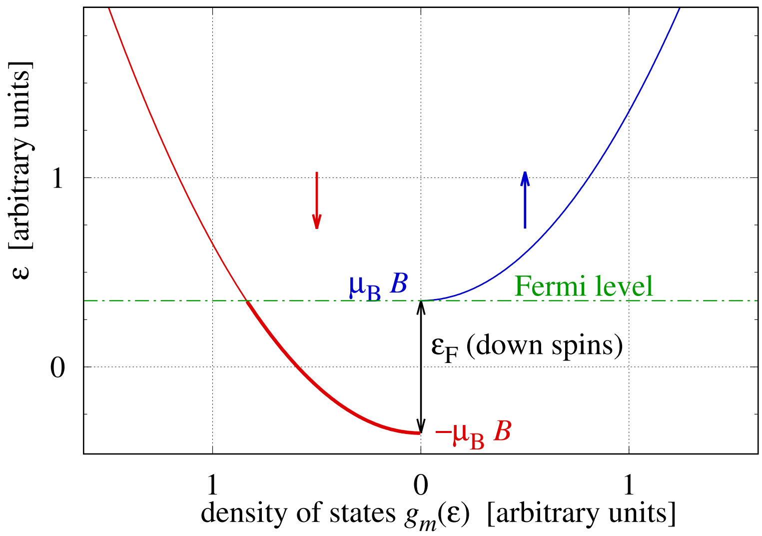
v) **(2.5 points)** Assume that (at a Hartree-Fock mean-field level) the electric repulsion between the electrons is compensated by their attraction to a uniform background of positive charge. In this effective independent-electrons picture, for the residual exchange effect of the electron-electron Coulomb interaction, take the following simplified Hamiltonian: $H_{\mathrm{exch}} = -\frac{1}{2}J(N_\uparrow^2 + N_\downarrow^2)^{1/2}$. Verify that, assuming $J > 0$, this term favors spin-aligned states relative to spin-antiparallel states. Estimate the value of J by the Coulomb repulsion of two uniformly delocalized electrons in the $L \times L \times L$ box.

vi) **(3 points)** In the continuum model of question ii), how does the presence of the exchange term H_{exch} modify the maximum number $N_{\max}$ of electrons that are 100% spin-aligned in the same magnetic field? To what value $N_{\max}^{\mathrm{exch}}$?

Solutions

(i) A magnetic field coupled purely to the electron spin lifts the spin-up energy levels by $2\,\frac{1}{2}\,\mu_{\rm B}\,B = \mu_{\rm B}\,B$, and decreases the spin-down energies by the same quantity.

The $T = 0$ Fermi gas is fully polarized as long as the Fermi level $\epsilon_{\rm F}$ strikes through the spin-down band but does not cross the spin-up band.



As illustrated in this figure, the maximum number of electrons compatible with 100% polarization occurs when the Fermi level sits infinitesimally below the bottom of the spin-up band. This condition is expressed by

$$2\mu_{\rm B}B = \epsilon_{\rm F}(\text{down spins})\,.$$

We obtain an equation for the corresponding maximum electron density, by noting that the spin degeneracy g_s for spin-down electron equals 1, rather than the usual 2 in the standard expression

$$\epsilon_{\rm F} = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2}{g_s}\,n\right)^{2/3}.$$

relating $\epsilon_{\rm F}$ to the number density n.

Accordingly:

$$2\mu_{\rm B}B = \frac{\hbar^2}{2m_e}\left(6\pi^2\,n\right)^{2/3},$$

$$2\mu_{\rm B}B\,\frac{2m_e}{\hbar^2} = \left(6\pi^2\,n\right)^{2/3},$$

$$\left(2\mu_{\rm B}B\,\frac{2m_e}{\hbar^2}\right)^{3/2} = 6\pi^2\,n\,,$$

$$n = \frac{1}{6\pi^2}\left(4\mu_{\rm B}B\,\frac{m_e}{\hbar^2}\right)^{3/2} = \frac{\left(4m_e\mu_{\rm B}B\right)^{3/2}}{6\pi^2\,\hbar^3}\,.$$

Numerically:

$$
\begin{aligned}
n &= \frac{(4m_e\mu_{\rm B}B)^{3/2}}{6\pi^2\,\hbar^3} = \frac{(2.9196\times 10^{-52}\ \mathrm{J\ kg})^{3/2}}{6.945\times 10^{-101}\ \mathrm{J^3\,s^3}} \\
&= \frac{4.98879\times 10^{-78}\ \ (\mathrm{J\ kg})^{3/2}}{6.945\times 10^{-101}\ \mathrm{J^3\,s^3}} = 7.183\times 10^{22}\ \mathrm{J^{-3/2}\,kg^{3/2}\,s^{-3}} \\
&= 7.183\times 10^{22}\ \mathrm{kg^{-3/2}\,m^{-3}\,s^3\,kg^{3/2}\,s^{-3}} \\
&= 7.183\times 10^{22}\ \mathrm{m^{-3}}.
\end{aligned}
$$

This is a relatively small density, at least 5 orders of magnitude smaller than regular conduction-electron densities of common metals. Such a small value is to be expected, given the smallness of magnetic energies compared to typical kinetic energies imposed by Pauli's principle at ordinary metallic densities.

(ii)

$$
N_{\rm max} = n\times L^3 = 7.183\times 10^{22}\ \mathrm{m^{-3}}\times 6.94265\times 10^{-23}\ \mathrm{m^3} = 4.9870\,,
$$

i.e. approximately 5 electrons.

(iii) In a cubic box with side L the $\vec{k}$ values allowed by the PBC are

$$
\vec{k} = \frac{2\pi}{L}\,(n_x,n_y,n_z)\,,\quad \text{with } n_\alpha = 0,\pm1,\pm2,....
$$

Correspondingly, the discrete kinetic-energy levels are

$$
E_{n_x,n_y,n_z} = \frac{\hbar^2}{2m_e}\,(n_x^2+n_y^2+n_z^2)\left(\frac{2\pi}{L}\right)^2 = \frac{(2\pi\hbar)^2}{2m_eL^2}\,(n_x^2+n_y^2+n_z^2)\,.
$$

Indicating the energy scale with $A=(2\pi\hbar)^2/(2m_eL^2)$, the resulting low-lying energy levels are:

n_x,n_y,n_z	degeneracy	energy
$0,0,0$	1	0
$\pm1,0,0$	6	A
$\pm1,\pm1,0$	12	$2A$
$\pm1,\pm1,\pm1$	8	$3A$
$\pm2,0,0$	6	$4A$
...		

With the given data, we compare the kinetic energy scale $A = 1.42662\times10^{-22}$ J $= 890.4\ \mu$eV with the magnetic one $2\mu_{\rm B}\,B = 1.60255\times 10^{-22}$ J $= 1000.2\ \mu$eV. We see that

$$
A < 2\mu_{\rm B}\,B < 2A\,.
$$

Therefore we obtain full spin polarization with enough electrons to fill the levels listed in the first 2 lines in the table (levels 0 and A) but not those in the third and the following lines (levels $2A$, $3A$, etc.). The resulting number of spin-down electrons is

$$
N_{\rm max}^{\rm PBC} = 1+6 = 7\,,
$$

visibly different from the result of Question (**ii**).

(iv) Replacing PBC with NBC, the wavefunctions switch from plane waves to real trigonometric functions. If we put the origin at one corner of the box, the free-electron wavefunctions compatible with NBC are proportional to products of $\sin(r_\alpha \pi n_\alpha/L)$, for each component $\alpha = x, y, z$. The "quantum numbers" $n_\alpha = 1, 2, 3, ...$, and the resulting energy levels are:

$$E_{n_x,n_y,n_z} = \frac{\hbar^2}{2m_e}\,(n_x^2+n_y^2+n_z^2)\left(\frac{\pi}{L}\right)^2 = \frac{(\pi\hbar)^2}{2m_e L^2}\,(n_x^2+n_y^2+n_z^2)\,.$$

We indicate the relevant energy scale with $A' = (\pi\hbar)^2/(2m_e L^2) \equiv A/4 = 3.5665 \times 10^{-23}$ J $=$ 222.6 μeV. The resulting low-lying energy levels are:

n_x, n_y, n_z	degeneracy	energy	excitation energy
$1,1,1$	1	$3A'$	0
$2,1,1$	3	$6A'$	$3A'$
$2,2,1$	3	$9A'$	$6A'$
$3,1,1$	3	$11A'$	$8A'$
$2,2,2$	1	$12A'$	$9A'$
$3,2,1$	6	$14A'$	$11A'$
...			

We see that

$$3A' < 2\mu_B\,B < 6A'\,.$$

Therefore we obtain full spin polarization with enough electrons to fill the levels listed in the first 2 lines in the table (levels $3A'$ and $6A'$) and leaving those above empty. The resulting number of spin-down electrons is

$$N_{\max}^{\rm NBC} = 1 + 3 = 4\,.$$

One more different value, compared to Questions 2 and 3!

(v) The spin-aligned states have $E^{\rm ferro} = -\frac{1}{2}JN$. States with different numbers $N_\downarrow, N_\uparrow \neq 0$ or N have larger energy because $N_\uparrow = N - N_\downarrow$, thus

$$\begin{aligned} E^{\rm exch} &= -\frac{J}{2}\left[(N-N_\downarrow)^2 + N_\downarrow^2\right]^{1/2} = -\frac{J}{2}\left[N^2 - 2N\,N_\downarrow + 2N_\downarrow^2\right]^{1/2} \\ &= -\frac{J}{2}\left[N^2 - 2N_\downarrow N_\uparrow\right]^{1/2} = -\frac{1}{2}JN\left[1 - 2\frac{N_\downarrow N_\uparrow}{N^2}\right]^{1/2} > -\frac{1}{2}JN, \end{aligned}$$

as long as both $N_\downarrow$ and $N_\uparrow$ are nonzero.

The number density of a uniformly delocalized electron over a volume V equals V^{-1}. The corresponding charge density is $-q_e V^{-1}$. The repulsion energy between two such uniform charge densities can be estimated by

$$
\begin{aligned}
J &= \int_V d^3r_1 \int_V d^3r_2 \left(\frac{q_e}{V}\right)^2 \frac{1}{4\pi\epsilon_0|\vec{r}_1-\vec{r}_2|} = \frac{q_e^2}{4\pi\epsilon_0 V^2}\int_V d^3r_1 \int_V d^3r_2 \frac{1}{|\vec{r}_1-\vec{r}_2|} \\
&= \frac{q_e^2}{4\pi\epsilon_0 V^2}\int_V d^3R \int_V d^3r \frac{1}{|\vec{r}|} = \frac{q_e^2}{4\pi\epsilon_0 V}\int_V d^3r \frac{1}{|\vec{r}|} \\
&= \frac{q_e^2}{4\pi\epsilon_0 V}\, 4\pi \int_0^{[3V/(4\pi)]^{1/3}} dr\, r^2 \frac{1}{r} = \frac{q_e^2}{4\pi\epsilon_0 V}\, 4\pi \int_0^{[3V/(4\pi)]^{1/3}} dr\, r \\
&= \frac{q_e^2}{4\pi\epsilon_0 V}\, 4\pi \left.\frac{r^2}{2}\right|_0^{[3V/(4\pi)]^{1/3}} = \frac{q_e^2}{4\pi\epsilon_0 V}\, 2\pi \left(\frac{3V}{4\pi}\right)^{2/3} = \frac{q_e^2}{4\pi\epsilon_0 V^{1/3}}\, 2\pi \left(\frac{3}{4\pi}\right)^{2/3} \\
&= \frac{q_e^2}{4\pi\epsilon_0 L}\, 3^{2/3} \left(\frac{\pi}{2}\right)^{1/3} = 1.357 \times 10^{-20}\ \mathrm{J} = 84715.7\ \mu\mathrm{eV}\,.
\end{aligned}
$$

(vi) The exchange term is minimum for fully aligned spins, e.g. $N_\downarrow = N$, $N_\uparrow = 0$, and it amounts to

$$E^{\text{ferro}} = -\frac{1}{2}JN.$$

The energy of a state with 1 flipped spin ($N_\downarrow = N-1$, $N_\uparrow = 1$) is

$$E^{1\text{ up}} = -\frac{J}{2}[(N-1)^2 + 1^2]^{1/2} = -\frac{J}{2}[N^2 - 2N + 2]^{1/2}.$$

The energy difference

$$
\begin{aligned}
\Delta E_{\text{exch}} = E^{1\text{ up}} - E^{\text{ferro}} &= -\frac{J}{2}[N^2-2N+2]^{1/2} + \frac{J}{2}N = \\
&= \frac{1}{2}JN\left[1 - \left(1 - \frac{2}{N} + \frac{2}{N^2}\right)^{1/2}\right] \\
&= \frac{1}{2}JN\left[1 - \left(1 + \frac{1}{2}\left(-\frac{2}{N}+\frac{2}{N^2}\right) - \frac{1}{8}\left(-\frac{2}{N}+\frac{2}{N^2}\right)^2 + O(N^{-3})\right)\right] \\
&= \frac{1}{2}JN\left[\frac{1}{2}\left(\frac{2}{N}-\frac{2}{N^2}\right) + \frac{1}{8}\left(-\frac{2}{N}+\frac{2}{N^2}\right)^2 + O(N^{-3})\right] \\
&= \frac{1}{2}JN\left[\frac{1}{N} - \frac{1}{N^2} + \frac{1}{8}\frac{4}{N^2} + O(N^{-3})\right] = \frac{1}{2}JN\left[\frac{1}{N} - \frac{1}{2N^2} + O(N^{-3})\right] \\
&= \frac{1}{2}J\left[1 - \frac{1}{2N} + O(N^{-2})\right]
\end{aligned}
$$

adds to the magnetic energy $2\mu_{\mathrm{B}}B$ required to flip a spin.

This exchange energy modifies the relation found for Question 2 to

$$2\mu_{\mathrm{B}}B + \Delta E_{\text{exch}} = \frac{\hbar^2}{2m_e}\left(6\pi^2\frac{N}{V}\right)^{2/3},$$

This equation is readily solved in the very-large-N limit, where one assumes $\Delta E_{\text{exch}} = \frac{1}{2}J$:

$$n = \frac{1}{6\pi^2\hbar^3}\left[(4\mu_{\text{B}}B + J)\, m_e\right]^{3/2}$$

Numerically:

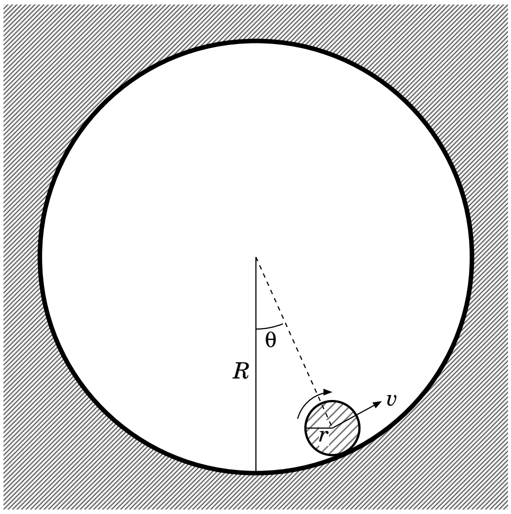
$$n = \frac{\left((4\mu_{\text{B}}B + J)\, m_e\right)^{3/2}}{6\pi^2\, \hbar^3} = \frac{\left(1.2656\times 10^{-50}\ \text{J kg}\right)^{3/2}}{6.945\times 10^{-101}\ \text{J}^3\,\text{s}^3}$$
$$= 2.050\times 10^{25}\ \text{J}^{-3/2}\,\text{kg}^{3/2}\,\text{s}^{-3} = 2.050\times 10^{25}\ \text{m}^{-3}.$$

This density is of course far larger than the result of Question 1. The corresponding maximum number of spin-aligned electrons in the same nano-volume $V = L^3$ is

$$N_{\text{max}}^{\text{exch}} = n\times L^3 = 2.050\times 10^{25}\ \text{m}^{-3}\times 6.94265\times 10^{-23}\ \text{m}^3 = 1423.3\,,$$

i.e. approximately 1423 electrons. Indeed $N = N_{\text{max}}^{\text{exch}} \gg 1$, which *a posteriori* justifies the large-N approximation in the calculation of ΔE_{exch}.

5 Classical Mechanics



A ball, with spherical symmetry, with external radius r, mass m, has a moment of inertia with respect to its center-of-mass (CM) equal to $I^* = \beta\, mr^2$. The ball rolls without slipping ("pure rolling") inside a spherical cavity of radius R. Everything is embedded in the gravitational field g at Earth's surface. Rolling frictions are negligible. The contact point of the ball on the spherical surface is displaced with respect to the lowest point by an angle θ, measured from the center of the spherical cavity.

Initially, the ball is located in the lowest point of the cavity and moves with a CM speed v_0.

In general, a pure rolling motion on a spherical surface has 2 degrees of freedom (DoF), that could be parameterized by a polar angle θ and an azimuthal angle ϕ.

i) **(2 points)** Write down the Lagrangian of the rolling ball as a function of θ, $\dot{\theta}$, ϕ, $\dot{\phi}$ and prove that in this specific problem only θ matters, i.e. that the problem has only 1 DoF.

(In case you fail proving the previous point, just assume it's true and continue...)

Write down, as functions of θ and of the other parameters of the problem, the following formulae:

ii) **(0.5 points)** the speed of the CM: $v(\theta) = \ldots$

iii) **(0.5 points)** the tangent acceleration of the CM: $a_t(\theta) = \ldots$

iv) **(0.5 points)** the reaction of the constraint, produced by the surface: $F_{constr}(\theta) = \ldots$

v) **(0.5 points)** the static friction, produced by the surface: $F_{frict}(\theta) = \ldots$

Keeping in mind the spherical symmetry of the rolling object:

vi) **(2 points)** what is the maximum possible value of β ($\beta_{\max}$)?

Now, let's assume $\beta = \beta_{\max}$, $r = 1$ cm, $R = 0.5$ m, $\mu_s = 0.9$:

vii) **(4 points)** what is the minimum value $v_0^{\min}$ of the initial speed v_0, such that the rolling ball keeps the conditions of "pure rolling" during all its motion? [provide both formula and numerical value]

Solutions

The CM of the rolling ball moves on a spherical surface of radius $(R-r)$, with speed v. Its height over the lowest position is $h=(R-r)(1-\cos\theta)$. The ball rolls with angular velocity ω, where the no-slipping condition imposes that $v=\omega r$; moreover, whatever the motion of the ball, its moment of inertia with respect to the rotation axis is always I^*, because of the spherical symmetry. Therefore, the kinetic energy of the rolling ball is

$$E_k \;=\; \frac{m}{2}v^2+\frac{I^*}{2}\omega^2 \;=\; \frac{m}{2}(1+\beta)v^2 \tag{51}$$

The potential energy can be written as

$$E_p \;=\; mgh \;=\; mg(R-r)(1-\cos\theta) \tag{52}$$

The velocity of the CM has two components:

$$v_\theta=(R-r)\dot\theta \quad ; \quad v_\phi=(R-r)\sin\theta\,\dot\phi$$

therefore the speed can be written as $v=(R-r)\sqrt{(\dot\theta)^2+(\sin\theta\,\dot\phi)^2}$. The Lagrangian is therefore

(i) $$\boxed{\mathcal{L} \;=\; E_k-E_p \;=\; \frac{m}{2}(1+\beta)(R-r)^2((\dot\theta)^2+(\sin\theta\,\dot\phi)^2)-mg(R-r)(1-\cos\theta)}$$

The Lagrangian is cyclical in ϕ (i.e. $\dfrac{d\mathcal{L}}{d\phi}=0$), therefore its conjugate momentum $L_\phi=\dfrac{d\mathcal{L}}{d\dot\phi}=m(1+\beta)[(R-r)\sin\theta]^2\dot\phi$ is a constant in the motion. Since it is null in the initial condition (when $\theta=0$), it remains null forever, which means that $\dot\phi=0$ and the motion is only along θ.

The forces acting on the rolling ball are:

- the weight $\vec F_g=-mg\,\hat z$, conservative, oriented downwards; it may be split along the directions $\hat n$, $\hat t$, respectively orthogonal and tangent to the surface, as:

$$\vec F_g=-mg(\cos\theta\,\hat n+\sin\theta\,\hat t)$$

- the reaction of the constraint $\vec F_{constr}=F_{constr}\,\hat n$, oriented towards the centre of the spherical cavity ($F_{constr}\geq 0$), not producing any work since the displacement is always orthogonal to it;

- the static friction $\vec F_{frict}=F_{frict}\,\hat t$, tangent to the surface, not producing any work since the contact point is always at rest; it must fulfil the condition:

$$|F_{frict}|\leq \mu_s\,F_{constr}$$

Questions **(ii)**, **(iii)** can be addressed using energy conservation:

$$E=E_k+E_p=\frac{m}{2}v^2+\frac{I^*}{2}\omega^2+mgh$$

From the condition of pure rolling, $\omega = \frac{v}{r}$, hence $I^*\omega^2 = \beta m r^2 \frac{v^2}{r^2} = m\beta v^2$. Moreover, the CM of the rolling ball moves along a circumference of radius $(R-r)$; choosing $h = 0$ for the CM in the lowest point of the cavity, we get $h = (R-r)(1-\cos\theta)$. Therefore we obtain:

$$E = \frac{m}{2}(1+\beta)v^2 + mg(R-r)(1-\cos\theta)$$

In the lowest point $E = \frac{m}{2}(1+\beta)v_0^2$, hence:

$$v^2 = v_0^2 - \frac{2g(R-r)(1-\cos\theta)}{1+\beta} \tag{53}$$

(ii) $\boxed{v(\theta) = \sqrt{v_0^2 - \frac{2g(R-r)(1-\cos\theta)}{1+\beta}}}$

Taking the derivative with respect to the time of the equation (53) we get:

$$2va_t = -\frac{2g(R-r)\sin\theta}{1+\beta}\frac{d\theta}{dt}$$

Now, the CM moves along a circumference of radius $(R-r)$ with angular speed $\frac{d\theta}{dt}$, hence $v = (R-r)\frac{d\theta}{dt}$. We can then work out (cancelling v away) that

(iii) $\boxed{a_t(\theta) = -\frac{g\sin\theta}{1+\beta}}$

Alternative approach to question **(iii)**: let's use the cardinal equation of the torques. Since the forces $\vec{F}_{constr}$, $\vec{F}_{frict}$ are adaptive — i.e. not calculable, before having already solved the motion — let's choose as a pole the contact point: the arms of both adaptive forces are then null, hence their torques are also null. On the opposite, the weight can be thought as if entirely applied to the CM; hence its torque has module $|M_g| = rmg\sin\theta$ and is oriented "outwards" from the plane of the drawing. Since the ball rolls clockwise (in the drawing), it has both $\vec{\omega}$ and $\vec{L}$ oriented "inwards" in the plane of the drawing, therefore it is convenient to choose as positive the "inwards" orientation. We get then:

$$L = I^*\omega + mrv = mr(1+\beta)v$$

and $M_g = -mrg\sin\theta$. From $M_g = \frac{dL}{dt}$ we get answer **(iii)** again.

Projecting all forces along $\hat{n}$ we get the centripetal force:

$$F_{constr} - mg\cos\theta = m\frac{v^2}{R-r}$$

from which — using equation (53) — we get

$$F_{constr} = m\left(\frac{v_0^2}{R-r} - \frac{2g(1-\cos\theta)}{1+\beta} + g\cos\theta\right)$$

(iv) $\boxed{F_{constr}(\theta) = \frac{mv_0^2}{R-r} + \frac{mg}{1+\beta}\left((3+\beta)\cos\theta - 2\right)}$

Projecting all forces along $\hat{t}$ we get the tangent force:

$$F_{frict} - mg\sin\theta = ma_t = -\frac{mg\sin\theta}{1+\beta}$$

(v) $\boxed{F_{frict}(\theta) = mg\sin\theta\,\frac{\beta}{1+\beta}}$

The maximal moment of inertia, for given r and m, is achieved when the mass is distributed such to maximize the distance from the rotation axis; for a spherical symmetry, this happens in the case of a hollow sphere, for which $I^* = \frac{2}{3}mr^2$. Therefore:

(vi) $\boxed{\beta_{\max} = \frac{2}{3}}$

The conditions to have a pure rolling are $F_{constr} > 0$ (contact) and $|F_{frict}| < \mu_s F_{constr}$ (no slipping). Obviously, the first is implied by the second, therefore we focus on that one. We must study the condition:

$$mg\sin\theta\frac{\beta}{1+\beta} < \mu_s\left(\frac{mv_0^2}{R-r} + \frac{mg}{1+\beta}((3+\beta)\cos\theta - 2)\right)$$

that, for $\beta = \frac{2}{3}$ (and cancelling m away) reads:

$$\frac{2}{5}g\sin\theta < \mu_s\frac{v_0^2}{R-r} + \frac{3}{5}\mu_s g\left(\frac{11}{3}\cos\theta - 2\right)$$

or

$$\underbrace{2\sin\theta - 11\mu_s\cos\theta}_{f(\theta)} < \mu_s\left(\frac{5v_0^2}{g(R-r)} - 6\right)$$

Let's now study the function $f(\theta)$:

$$f'(\theta) = 2\cos\theta + 11\mu_s\sin\theta \quad ; \quad f''(\theta) = -f(\theta)$$

The extremals are obtained by setting $f'(\theta_0) = 0$, which yields $\tan(\theta_0) = -\frac{2}{11\mu_s} = -0.202$. In the interval of interest, $\theta \in [0;\pi]$, the only solution is $\theta_0 = 2.942$ rad; in that condition, $f(\theta_0) = 10.10$ and $f''(\theta_0) = -f(\theta_0) < 0$, therefore this is a maximum. Therefore, if $\mu_s\left(\frac{5v_0^2}{g(R-r)} - 6\right) > f(\theta_0)$ we are sure that pure rolling always holds. In terms of v_0, this conditions reads:

$$v_0^2 > \frac{g(R-r)}{5}\left(\frac{f(\theta_0)}{\mu_s} + 6\right)$$

i.e.:

(vii) $\boxed{v_0^{\min} = \sqrt{\frac{g(R-r)}{5}\left(\frac{f(\theta_0)}{\mu_s} + 6\right)} = 4.07 \text{ m/s}}$

6 Electromagnetism

A coaxial cable (Figure 4a) consists of a central *ohmic* conductor of radius a, length h, resistance R surrounded by a coaxial perfectly conducting cylinder of radius b; the region between the central conductor and the external cylinder is free space.

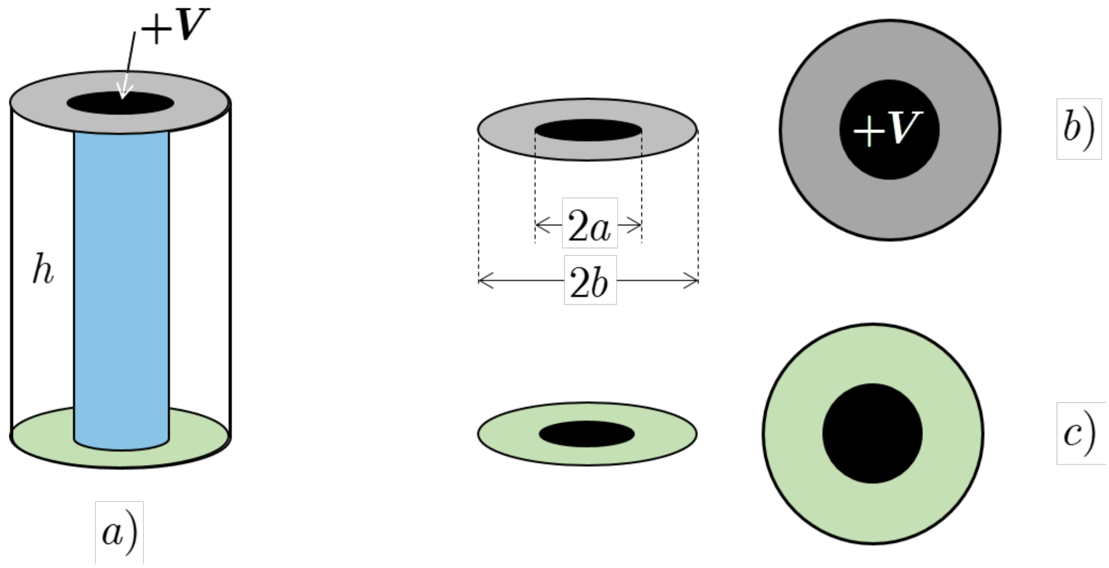


Figure 4: geometry of the battery a), coaxial cable b), and resistor c)

The cable is connected from on one side to a circular battery of negligible thickness (Figure 4b); the region $r < a$ is at potential V, the potential inside the battery decreasing radially to zero as

$$\phi(r) = \frac{V}{\ln\frac{a}{b}} \ln\frac{r}{b}$$

On the other side, the cable is connected to a circular load resistor R_L, constructed with *ohmic* conductor, of negligible thickness, of inner and outer radii a and b respectively (Figure 4c). Consider the four regions

region 1	$0 < r < a$ $0 < z < h$	region 2	$a < r < b$ $z = 0$
region 3	$a < r < b$ $z = h$	region 4	$a < r < b$ $0 < z < h$

i) **(4 points)** Determine the electric field in each of the four regions and the surface charge densities on the boundaries separating the four regions. *Hint:* solve the Laplace equation in cylindrical coordinates (see also the first hint of problem 1) using variables separation method with separation constant equal to 0.

ii) **(2 points)** Determine the Poynting vector in the above regions and discuss the energy fluxes among the various regions.

iii) **(4 points)** Finally, assume $R = 0$ and determine the electromagnetic momentum of the system and compare it with movements of the parts of the system, if any.

Solutions

(i) Electric Field in regions 1, 2, 3, 4

The problem of the electric field associated to a conductor carrying a steady current has not received much attention in textbooks; notable exceptions are Sommerfeld [1], Jefimenko [2], Haus and Melcher [3]. The problem has received attention in literature. The problem presented here is based on the treatment of Marcus [4], Russell [5], McDonald [6]. The symmetry of the problem suggests the use of a cylindrical coordinate system with the z axis along the axis of the inner conductor; because of the azimuthal symmetry, the potential is function only of the z and r coordinates $\phi(r,z)$. Problems of steady currents in *ohmic* conductors are usually solved with Laplace equation, possibly with mixed boundary conditions. The boundary conditions of this problem (that will be stated in the following) lead to the general solution, obtained variables separation method with separation constant equal to 0 [4], [5]

$$\phi(r,z) = (A + B\ln r)(e + fz)$$

The total current in the conductors is $I = V/(R + R_L)$; the voltage drop in the inner conductor is $VR/(R + R_L)$, the remaining voltage $VR_L/(R + R_L$ appears across the load resistor. For future convenience we define $\alpha = R/(R + R_L)$, $\beta = R_L/(R + R_L)$, $\alpha + \beta = 1$.

Electric Field inside the inner conductor, region 1

Since for $r > a$ there is no conduction, the current density (and the electric field) do not have component normal to the wire surface. The boundary conditions are

$$\phi_1(r,h) = V \qquad \phi_1(r,0) = \tfrac{R_L}{R+R_L}V = \beta V \qquad \tfrac{\partial \phi_1}{\partial r}\Big|_{r=a} = 0$$

Application of the boundary conditions lead to the solution in region 1

$$\phi_1(r,h) = V(\beta + \alpha\tfrac{z}{h}) \qquad \mathbf{E}_1 = -\tfrac{\alpha V}{h}\hat{\mathbf{e}}_z \qquad \begin{array}{c} 0 < r < a \\ 0 < z < h \end{array}$$

Electric Field in the load resistor, region 2

The load resistor has negligible thickness and the current assumes the form of a surface current density $\mathbf{K}(r) = 1/\eta\mathbf{E}(r)$, η being the surface resistivity. The zero thickness remove the z dependence in the potential; the boundary conditions are

$$\phi_2(r=a) = V\tfrac{R_L}{R+R_L} = \beta V \qquad \phi_2(r=b) = 0$$

Application of the boundary conditions lead to the solution in region 2

$$\phi_2(r) = V\tfrac{R_L}{R+R_L}\tfrac{1}{\ln\frac{a}{b}}\ln\tfrac{r}{b} = \tfrac{\beta V}{\ln\frac{a}{b}}\ln\tfrac{r}{b} \qquad \mathbf{E}_2 = -\tfrac{\beta V}{\ln\frac{a}{b}}\tfrac{1}{r}\hat{\mathbf{e}}_r = \tfrac{\beta V}{\ln\frac{b}{a}}\tfrac{1}{r}\hat{\mathbf{e}}_r \qquad \begin{array}{c} a < r < b \\ z = 0 \end{array}$$

Electric field in the battery, region 3

The potential distribution in the battery is given; the resulting electric field is

$$\phi_3(r) = \frac{V}{\ln\frac{a}{b}} \ln\frac{r}{b} \qquad \mathbf{E}_3 = -\frac{V}{\ln\frac{a}{b}}\frac{1}{r}\hat{\mathbf{e}}_r = \frac{V}{\ln\frac{b}{a}}\frac{1}{r}\hat{\mathbf{e}}_r \qquad \begin{matrix} a < r < b \\ z = h \end{matrix}$$

Notice that the current flows in the opposite direction of the electric field because the battery provides the necessary electromotive force to maintain the current.

Electric Field in the free space region 4

In the free space region the potential must be 0 at the outer boundary $r = b$ and satisfy the potential continuity condition at the boundaries with regions 1, 2, 3. The resulting potential is

$$\phi_4(r,z) = \frac{V}{\ln\frac{a}{b}} \ln\frac{r}{b}\left(\beta + \alpha\frac{z}{h}\right) \qquad \mathbf{E}_4 = -\frac{V}{\ln\frac{a}{b}}\frac{1}{r}\left(\beta + \alpha\frac{z}{h}\right)\hat{\mathbf{e}}_r - \frac{V}{\ln\frac{a}{b}}\frac{\alpha}{h}\ln\frac{r}{b}\hat{\mathbf{e}}_z \qquad \begin{matrix} a < r < b \\ 0 < z < h \end{matrix}$$

Electric field plot and surface charge densities

A qualitative plot of the electric field is shown in Figure 5. The component of the electric field in region 4 normal to surfaces that limit the free space region determines the surface charge densities on the boundaries.

boundary $1-4$ $\qquad \sigma_1(z) = \varepsilon_0 E_{4r}(a,z) = -\frac{\varepsilon_0 V}{\ln\frac{a}{b}}\frac{1}{a}\left(\beta + \alpha\frac{z}{h}\right)$

boundary 4 $\qquad \sigma_4(z) = -\varepsilon_0 E_{4r}(b,z) = \frac{\varepsilon_0 V}{\ln\frac{a}{b}}\frac{1}{b}\left(\beta + \alpha\frac{z}{h}\right)$

boundary $4-3$ $\qquad \sigma_3(r) = \varepsilon_0 E_{4z}(r,h) = +\frac{\varepsilon_0 V}{\ln\frac{a}{b}}\frac{\alpha}{h}\ln\frac{r}{b}$

boundary $4-2$ $\qquad \sigma_2(r) = -\varepsilon_0 E_{4z}(r,0) = -\frac{\varepsilon_0 V}{\ln\frac{a}{b}}\frac{\alpha}{h}\ln\frac{r}{b}$

Please notice $\ln a/b < 0$ in the above formulas.

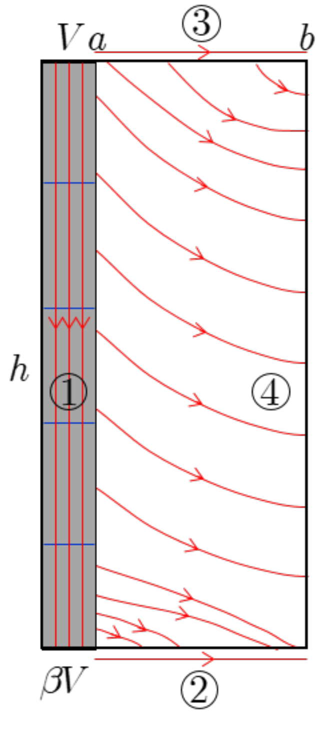


Figure 5: Qualitative plot of the electric field in the four regions

(ii) Poynting Vector and Power Flux

The steady current condition allows to use the Ampere's circuital law without the displacement current term. Due to the problem symmetry, the magnetic induction field lines are circumferences coaxial with the z axis. We obtain

$$\mathbf{B} = -\tfrac{\mu_0 I}{2\pi}\tfrac{1}{r}\hat{\mathbf{e}}_\phi \qquad a < r < b$$

$$\mathbf{B} = -\tfrac{\mu_0 I}{2\pi}\tfrac{r}{a^2}\hat{\mathbf{e}}_\phi \qquad 0 < r < a$$

The Poynting vector in region 4 is simply

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = (E_z\hat{\mathbf{e}}_z + E_r\hat{\mathbf{e}}_r)\times(B_\phi\hat{\mathbf{e}}_\phi) = -\frac{E_z B_\phi}{\mu_0}\hat{\mathbf{e}}_r + \frac{E_r B_\phi}{\mu_0}\hat{\mathbf{e}}_z$$

Inserting the expression of the electric field calculated previously we obtain

$$\mathbf{S} = \frac{VI}{2\pi}\left[-\frac{\ln r/b}{\ln a/b}\frac{\alpha}{h}\frac{1}{r}\hat{\mathbf{e}}_r + (\beta + \alpha\frac{z}{h})\frac{1}{r^2\ln a/b}\hat{\mathbf{e}}_z\right]$$

The Poynting vector $\mathbf{S}$ has radial and axial components; is zero on the other surface ($r = b$). We now evaluate the flux on the surfaces of the battery, of the central conductor and of the load resistor. Let's start with the flux on the battery surface

$$\Phi_3 = \int_{\text{battery}} \mathbf{S}\cdot d\mathbf{a} = \int_a^b \frac{VI}{2\pi}\left[(\beta + \alpha\frac{z}{h})\frac{1}{r^2\ln a/b}\hat{\mathbf{e}}_z\right]_{z=h}(-\hat{\mathbf{e}}_z)2\pi r dr = -VI\int_a^b \frac{dr}{r\ln a/b} = VI$$

that is equal to the power supplied by the battery, as expected.
The flux on the central conductor

$$\Phi_1 = \int_{\text{inner}} \mathbf{S}\cdot d\mathbf{a} = \int_0^h \left[-\frac{\ln r/b}{\ln a/b}\frac{\alpha}{h}\frac{1}{r}\hat{\mathbf{e}}_r\right]_{r=a}\cdot\hat{\mathbf{e}}_r 2\pi a dz = -\alpha VI = -\frac{R}{R+R_L}VI = -RI^2$$

Again, this is the power dissipated in the central conductor.

Finally, the flux on the load resistor is

$$\Phi_2 = \int_{\text{load}} \mathbf{S}\cdot d\mathbf{a} = \int_a^b \frac{VI}{2\pi}\left[(\beta + \alpha\frac{z}{h})\frac{1}{r^2\ln a/b}\hat{\mathbf{e}}_z\right]_{z=0}\cdot\hat{\mathbf{e}}_z 2\pi r dr = \frac{VI\beta}{\ln a/b}\int_a^b \frac{dr}{r} = -\beta VI = -R_L I^2$$

as expected.

(iii) Momentum of the electromagnetic field

The momentum density $\mathbf{g}$ of the electromagnetic field is proportional to the Poynting vector; in region 4 we have

$$\mathbf{g} = \frac{\mathbf{S}}{c^2} = \frac{VI}{2\pi c^2}\left[-\frac{\ln r/b}{\ln a/b}\frac{\alpha}{h}\frac{1}{r}\hat{\mathbf{e}}_r + (\beta + \alpha\frac{z}{h})\frac{1}{r^2\ln a/b}\hat{\mathbf{e}}_z\right]$$

The total momentum of the system is obtained by integrating the density $\mathbf{g}$ over the volume[1]

$$\mathbf{p}_{EM} = \int_{\text{cable}} \mathbf{g} dV = \frac{VI}{2\pi c^2}\frac{\hat{\mathbf{e}}_z}{\ln a/b}\int_0^h (\beta + \alpha\frac{z}{h})dz \int_a^b \frac{2\pi r}{r^2}dr = -\frac{VIh}{c^2}(\beta + \frac{\alpha}{2})\hat{\mathbf{e}}_z$$

At first sight, it is certainly strange to find a non zero momentum in a system with time independent electromagnetic fields; this has been discussed by Griffiths [7] and more recently by Boyer [8], McDonald [6] and Babson et al. [9]. The relevant point here is that there is energy that leaves the battery ($z_{battery} = h$) and increases the energy content of the load (Joule dissipation, at $z_{load} = 0$). To eliminate Joule dissipation in the inner conductor, for the rest of the discussion let's assume that the central conductor has zero resistance ($R = 0$). The electromagnetic momentum become

$$\mathbf{p}_{EM} = -\frac{I^2 h}{c^2}R_L\hat{\mathbf{e}}_z$$

Assuming that the system is isolated, the center of energy of the system must be at rest. The coordinate of the center of energy z_{CE} is

$$z_{CE} = \frac{1}{M_T c^2}\int_0^h z\left[\rho_{Matter}(z)c^2 + \rho_{Energy}(z)\right]dz \equiv z_{CE\ Matter} + z_{CE\ Energy}$$

In the above definition ρ_{Matter} and ρ_{Energy} are, respectively, the matter density and the energy density initially contained the battery, and $M_T c^2 = M_{Matter}c^2 + U$ is the total energy of the system [2]. Since the center of energy is at rest we have

$$\frac{dz_{CE}}{dt} = 0 \qquad \frac{dz_{CE\ Matter}}{dt} + \frac{dz_{CE\ Energy}}{dt} = 0 \qquad v_{CE\ Matter} = -\frac{dz_{CE\ Energy}}{dt} = 0$$

Finally we have

$$dz_{CE\ Energy} = \frac{1}{M_T c^2}(z_{Load}Pdt - z_{Battery}Pdt) = -\frac{hPdt}{M_T c^2} \qquad v_{CE\ Matter} = -\frac{dz_{CE\ Energy}}{dt} = \frac{hVI}{M_T c^2}$$

We conclude that the electromagnetic momentum $\mathbf{p}_{EM}$ is compensated by an equal and opposite momentum of the matter $M_T v_{CEMatter}$. The origin of the matter momentum is in the forces that act on the cable during the initial transient, when the battery is switched on, before the steady condition is reached. A detailed analysis of the transient need further work. The recoil speed of the matter is of order $1/c^2$ and it is certainly not measurable; however, it is interesting to observe the self-consistency of the electromagnetic theory.

[1] We should have included also the momentum density inside the central conductor; however this integrates to zero

[2] We do not consider the contribution of electromagnetic energy stored in the free space between the inner and outer conductor: this energy is negligible and time independent

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7 Medical Physics

In conventional external beam radiation therapy, X-rays are used to impart a radiation dose to the tumour, sparing the surrounding healthy tissue as much as possible. Consider a medical linear accelerator (linac) where electrons are accelerated to an energy of 15 MeV and then stopped in a metal target. A portion of their kinetic energy is transformed into bremsstrahlung X-rays that form a radiotherapy photon beam. Because of the energies of the produced X-rays and the low atomic number of tissues, the most important interaction of the photon beam with patient tissues is Compton scattering.
Consider a Compton effect interaction where a photon is scattered on a free (i.e. loosely bound) electron with a scattering angle θ (i.e. angle between the incident photon direction and the scattered photon direction).

i) **(1 point)** Derive the expression of the kinetic energy of the recoil electron as a function of the incident photon energy and of the scattering angle θ;

ii) **(1 point)** Show that the energy of photon scattered with angles θ larger than $\pi/2$ cannot exceed 511 keV no matter how high the incident photon energy is.

Consider now the interaction of the X-rays produced by the linac with an absorber, and the related interaction coefficients μ/ρ (mass attenuation coefficient) and μ_{en}/ρ (mass energy absorption coefficient).

iii) **(1 point)** Express the ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ in terms of: (a) the mean fraction f_j of the incident photon energy that is transferred to kinetic energy of charged particles in an interaction of type j, (b) the component cross section σ_j relating to the interaction of type j, and (c) the fraction g of the kinetic energy transferred by photons to charged particles that is subsequently lost in radiative processes.

The ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ vs. photon beam energy produced by the linac, for a high Z absorber (lead), is plotted in Figure 6.

iv) **(1 point)** How do you explain the minimum value of the ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ in lead around 100 keV?

v) **(1 point)** How do you explain the decrease of the ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ in lead with increasing the photon energy occurring above approximately 10 MeV?

Accelerated charged particles, with mass $m \gg m_{electron}$, can also be used for external beam radiotherapy to treat tumours. The advantage of hadrontherapy localised dose deposition with respect to conventional radiotherapy is related to the typical energy release of such particles in matter, which follows the Bethe-Bloch equation. The presence of the Bragg peak, that appears when the heavy charged particle velocity approaches to zero, defines also the concept of *range* of such particles in matter.

vi) **(1 point)** Describe the main dependencies of hadrons energy loss in matter, discussing the characteristics of the Bragg peak (why it is a peak and why it has a non-null width).

vii) **(1 point)** Express the range scaling laws in the case of:

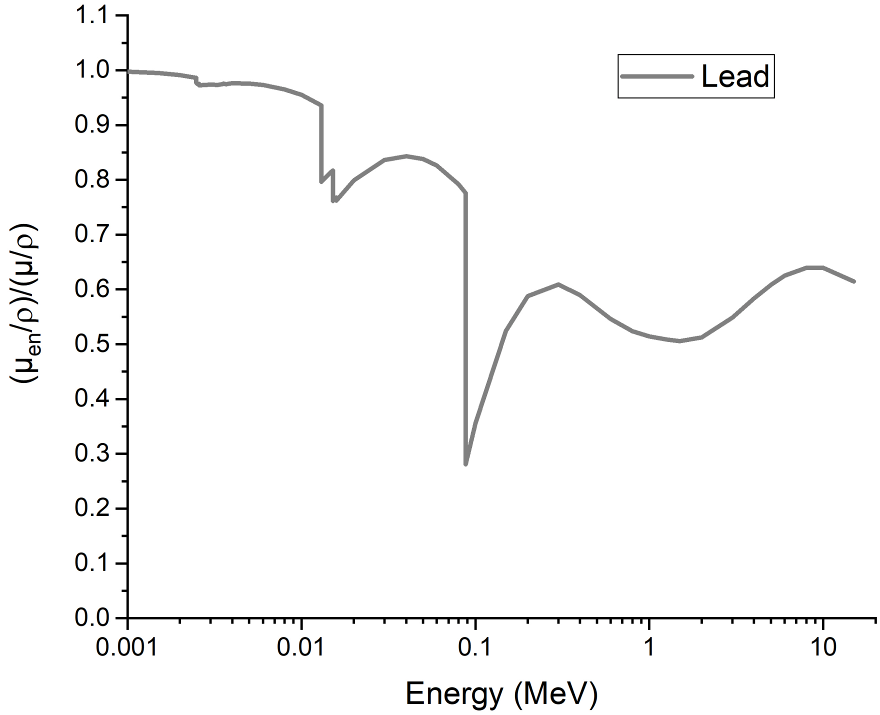


Figure 6

a) different particles with same velocity, traversing the same medium;

b) same particle traversing different media.

viii) **(1 point)** Compute the ratio of protons range with respect to carbon ions range, when protons and carbon ions have the same velocity.

Charged particles undergo also multiple coulomb scattering (MCS) by the nuclei field when traversing a medium, causing small deflections from the primary track path.

ix) **(1 point)** Describe the expression of the root mean squared deflection angle projected to a plane (θ_0) due to the MCS in the case of small traversed thickness.

x) **(1 point)** Considering a 150 MeV proton and a 285 MeV/u carbon ion, compute the ratio of θ_0 of proton over carbon ion after 1 meter of air.

Solutions

(i) The relativistic conservation of total energy and momentum laws are used in the derivation of the expression of the kinetic energy of the recoil electron.
The conservation of total energy is expressed as:

$$h\nu + m_e c^2 = h\nu' + E_e = h\nu' + m_e c^2 + E_K, \tag{54}$$

where $h\nu$ is the incident photon energy, $h\nu'$ is the scattered photon energy, $m_e c^2$ is the rest energy of the recoil electron, E_e is the total energy of the recoil electron and E_K is the kinetic energy of the recoil electron.
The conservation of momentum, together with trigonometric laws, can be expressed as:

$$p_e^2 = p^2 + (p')^2 - 2pp' \cos\theta, \tag{55}$$

where p_e is the momentum of the recoil electron, p is the momentum of the incident photon and p' is the momentum of the scattered photon.
By properly combining the two above expressions, the kinetic energy of the recoil electron can be derived as:

$$E_K = h\nu \cdot \frac{\alpha \cdot (1-\cos\theta)}{1+\alpha\cdot(1-\cos\theta)}, \tag{56}$$

where $\alpha = \frac{h\nu}{m_e c^2}$.

(ii) The energy of the scattered photon may be derived as:

$$h\nu' = h\nu \cdot \frac{1}{1+\alpha\cdot(1-\cos\theta)}. \tag{57}$$

For a given angle θ, the scattered photon energy increases with increasing the incident photon energy up to a saturation value.
For $\theta = \pi/2$ the scattered photon energy is equal to $h\nu' = \frac{h\nu}{1+\frac{h\nu}{m_e c^2}}$ characterized by a saturation value of $h\nu' \to m_e c^2 = 511\,\mathrm{keV}$ for $h\nu \to \infty$.
For larger angles, the saturation value of the scattered photon energy is lower than $511\,\mathrm{keV}$ and for $\theta = \pi$ the saturation value of the scattered photon energy is equal $\frac{1}{2} m_e c^2$.

(iii) The final expression to be reported is:

$$\frac{\mu_{en}/\rho}{\mu/\rho} = \sum_j \frac{f_j \sigma_j}{\sigma_j} \cdot (1-g). \tag{58}$$

Indeed, it should be recognized that:

$$(\mu_{en}/\rho) = (\mu_{tr}/\rho)(1-g), \tag{59}$$

where (μ_{tr}/ρ) is the mass energy transfer coefficient that can be written as:

$$(\mu_{tr}/\rho) = (\mu/\rho) \cdot f_{tr} = (\mu/\rho) \cdot \sum_j \frac{f_j \sigma_j}{\sigma_j}, \tag{60}$$

where f_{tr} is the total mean energy transfer fraction. Therefore:

$$(\mu_{en}/\rho) = (\mu/\rho) \cdot \sum_j \frac{f_j \sigma_j}{\sigma_j} \cdot (1 - g), \tag{61}$$

and:

$$\frac{\mu_{en}/\rho}{\mu/\rho} = \sum_j \frac{f_j \sigma_j}{\sigma_j} \cdot (1 - g). \tag{62}$$

Basically, the ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ corresponds to the total mean energy absorption fraction:

$$f_{ab} = \frac{\mu_{en}/\rho}{\mu/\rho} = f_{tr} \cdot (1 - g). \tag{63}$$

(iv) In the energy range we are considering, the minimum value of the ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ essentially corresponds to the minimum value of the total mean energy transfer function f_{tr}. For high atomic number Z absorbers such as lead, f_{tr} has a minimum value when the photon energy equals the K shell binding energy.

(v) The decrease of the ratio $\frac{\mu_{en}/\rho}{\mu/\rho}$ (i.e. the decrease of the total mean energy absorption fraction f_{ab}) with increasing the photon energy is due to the fact that, for a given absorber material, the mean radiation fraction g increases with photon energy, mainly by bremsstrahlung interaction of the secondary charged particles while they travel through the absorbing medium. While f_{tr} continues to increase with increasing the photon energy, f_{ab} attains a local maximum and then decrease as result of the increase of g. The energy at which the local peak in f_{ab} occurs is inversely proportional to the absorber atomic number Z appearing at approximately 10 MeV for high Z absorbers, as lead.

(vi) The energy loss of charged hadrons in matter is described by the Bethe-Bloch formula,

$$\frac{dE}{dx} \propto \frac{z^2}{\beta^2} \rho \frac{Z}{A} \left[\ln \left(\frac{\dots}{I^2} \right) + \dots - 2\frac{C}{Z} - \delta \right], \tag{64}$$

where:

- ρ is the material density;
- Z/A is the material ratio of the atomic number over mass number (which is always ~ 0.5 except for Hydrogen);
- I is the material mean excitation energy;
- z is the impinging particle atomic number;
- $\beta = v/c$ is the impinging particle velocity;
- C is the shell correction, important at low energy;

- δ is the density correction, important at high energies.

The energy loss of charged hadrons in matter as a function of the traversed depth is characterized by the Bragg curve: at the material entry channel, the particle has its maximum velocity and therefore lower energy loss. With increasing depth, the particle velocity decreases and the energy loss increases. At the end of the particle trajectory (β 10^{-2}), the particle spends a "long" time in proximity of any material electron, transferring a large amount of energy. Then it starts to pick up the electrons, lowering its effective charge (z) and the stopping power drops, resulting in a peak, called Bragg peak.
The width of the peak depends on the stochastic nature of the Columbian interactions of charged hadrons with material nuclei: the number of interactions and the amount of kinetic energy transferred to the atomic electrons at each interaction is different for each particle (with same beta traversing the same medium), causing the energy loss statistical fluctuations.
The Bragg peak position depends on the charged hadron initial kinetic energy: the more the energy, the deeper the Bragg peak position.

(vii) The range of the charged hadron can be derived by the Bethe-Bloch formula.
For different particles traversing the same material, R can be expressed as a function of β:

$$R = \int_0^R dx = \int_0^R \frac{dx}{dE} dE = \int_0^{E_0} \frac{dE}{z^2 f_E(E)} = \int_0^{\beta_0} \frac{\beta d\beta}{f_\beta (1-\beta^2)^{3/2}}. \tag{65}$$

For particles with same β, traversing different materials, R can be expressed by the Bragg-Kleeman rule:

$$R = \alpha E^p, \tag{66}$$

with

$$\alpha = c\frac{\sqrt{A}}{\rho}. \tag{67}$$

a) For different particles traversing the same material, with same β, applying Eq. 65:

$$\frac{R_a(\beta)}{R_b(\beta)} = \frac{m_a z_b^2}{m_b z_a^2}. \tag{68}$$

b) For the same particle traversing different materials, applying Eq. 66:

$$\frac{R_1}{R_2} = \frac{\rho_2}{\rho_1} \cdot \frac{\sqrt{A_1}}{\sqrt{A_2}}. \tag{69}$$

(viii) Applying Eq. 68:

$$R_p = \frac{m_p z_{12C}^2}{12 m_p z_p^2} R_{12C} = \frac{36}{12} R_{12C} = 3 \cdot R_{12C}. \tag{70}$$

(ix) The expression of the root mean squared deflection angle is:

$$\theta_0 = \frac{13.6\,\mathrm{MeV}}{\beta c p} z \sqrt{\frac{\Delta x}{X_0}} \left(1 + 0.038 \ln(\frac{\Delta x}{X_0})\right), \tag{71}$$

where:

- $\beta = v/c$ of the particle projectile;
- c is the light velocity;
- p is the particle momentum;
- z is the particle charge (atomic number);
- Δx is the traversed thickness;
- X_0 is the radiation length, which is equal to 7/9 of the mean free path for pair production by a high energy photon.

θ_0 is lower for heavy particles and/or particles with high kinetic energy.
This formula is valid for small material thickness or very low density materials, where the energy loss due to MCS is negligible.

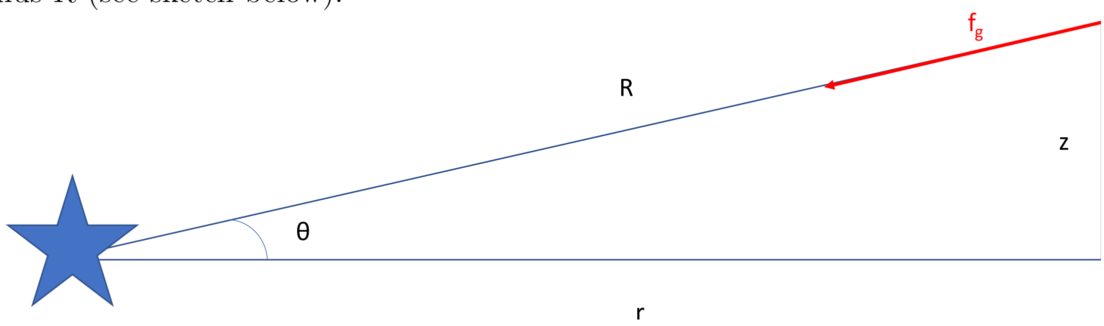
(x) Since air is a very low density material, applying the formula at point (ix):

$$\frac{\theta_{0p}}{\theta_{012C}} = \left(\frac{z_p}{(\beta cp)_p}\right)\left(\frac{(\beta cp)_{12C}}{z_{12C}}\right) \sim \left(\frac{z_p}{(\beta cp)_p}\right)\left(\frac{2(\beta cp)_p}{6z_p}\right) = 3 \tag{72}$$

8 Astrophysics of protoplanetary discs

Stars form from gas gravitational collapse in molecular clouds. As a result of angular momentum conservation, the collapse also produces discs, namely dense structures concentrated around a specific plane (often called the disc mid-plane), which rotate around the newly formed star. It is in these discs that planets form and for this reason they are often called protoplanetary discs. Given the geometry of these systems, it is convenient to use a cylindrical coordinate system to describe them.

IMPORTANT: pay attention to the difference between cylindrical radius r and spherical radius R (see sketch below).



i) **(3 points)** Assume for simplicity that the disc is vertically isothermal. In this case the gas pressure P is simply given by $P = c_s^2 \rho$, where c_s is the gas sound speed and ρ is the gas density. After further assuming that the gas is in hydrostatic equilibrium and so there is no motion along the vertical direction, derive the vertical density profile $\rho(z)$. To do so you can neglect the disc self-gravity and assume that only the star contributes to the gravitational force.

ii) **(1 point)** These discs are typically geometrically thin, meaning that their thickness H is much smaller than the radial coordinate r: $H \ll r$. Simplify the expression you derived in the previous question under this hypothesis and derive a relation between the gas density in the midplane $\rho_0(r) = \rho(r, z = 0)$ and the disc surface density, defined as $\Sigma = \int_{-\infty}^{+\infty} \rho(z) \mathrm{d}z$, that is, the vertical integral of the disc density.

iii) **(3 points)** Considering only the disc midplane at $z = 0$, derive the rotation curve $v_\phi(r)$ of the disc. As before, you can neglect the disc self-gravity and assume that only the star contributes to the gravitational field. In addition to gravity, you need to take into account gas pressure. For simplicity, you can assume that the gas density in the midplane is a power-law function of the radial coordinate: $\rho_0 \propto r^{-p}$. In the same way, you can also assume that the sound speed $c_s(r) \propto r^{-q}$ is a power-law.

iv) **(1 point)** Proto-planetary discs are close to Keplerian rotation. Using the expression you derived in the previous point, quantify the fractional deviation of the rotation curve from Keplerian rotation, and show that this is indeed small provided that the disc is thin.

v) **(2 points)** Generalise the expression you derived for the rotation curve for $z \neq 0$. As in question 2, use the thin disc approximation and only consider the case of small z/r. Does gas at high altitude rotate faster or slower than the gas in the midplane?

Hints:
The operator $(\mathbf{A}\cdot\nabla)\mathbf{B}$ in cylindrical coordinates is the following:

$$
\begin{aligned}
(\mathbf{A}\cdot\nabla)\mathbf{B} = & \left(A_r \frac{\partial B_r}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_r}{\partial \phi} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r}\right)\hat{\mathbf{r}} + \\
& + \left(A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_\phi}{\partial \phi} + A_z \frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r}\right)\hat{\phi} + \\
& + \left(A_r \frac{\partial B_z}{\partial r} + \frac{A_\phi}{r}\frac{\partial B_z}{\partial \phi} + A_z \frac{\partial B_z}{\partial z}\right)\hat{\mathbf{z}}
\end{aligned}
$$

Solutions

(i) To get the density profile we need to solve for hydrostatic vertical equilibrium in the vertical direction:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = f_{\mathrm{grav},z} = -\frac{GM_* z}{(r^2+z^2)^{3/2}}. \tag{73}$$

Substituting $P = \rho c_s^2$ and taking c_s vertically constant:

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} = -\frac{GM_* z}{c_s^2 (r^2+z^2)^{3/2}}, \tag{74}$$

which is separable and can be directly integrated:

$$\log\rho - \log\rho_0 = -\frac{GM_*}{c_s^2}\left(\frac{1}{r} - \frac{1}{R}\right) \tag{75}$$

to give the final expression:

$$\rho = \rho_0 \exp\left[\frac{GM_*}{c_s^2}\left(\frac{1}{R} - \frac{1}{r}\right)\right]. \tag{76}$$

(ii) Consider the factor

$$\frac{1}{R} - \frac{1}{r} = \frac{1}{(r^2+z^2)^{1/2}} - \frac{1}{r}. \tag{77}$$

We can easily expand this to first order in z^2/r^2 to

$$\frac{1}{R} - \frac{1}{r} \simeq -\frac{z^2}{2r^3}, \tag{78}$$

which implies that the density reduces to

$$\rho = \rho_0 \exp\left(-\frac{GM_*}{c_s^2 r^3}\frac{z^2}{2}\right). \tag{79}$$

We can write this as

$$\rho = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right), \tag{80}$$

which makes it clear that the density profile is a Gaussian function. The scale-height H of the Gaussian is given by

$$H = \frac{c_s r^{3/2}}{(GM_*)^{1/2}} = \frac{c_s}{\Omega_K}, \tag{81}$$

which we have simplified recognising that $\sqrt{\frac{GM_*}{r^3}}$ is the Keplerian angular frequency.

The surface density profile can be obtained from the integration along $\hat{z}$:

$$\Sigma(r) = \int_{-\infty}^{\infty} \rho_(z,r)\mathrm{d}z\ .$$

If we insert the expression of ρ found before we get:

$$\Sigma(r) = \rho_0 H \sqrt{2\pi} \ .$$

(iii) To get the velocity profile we need to write force balance in the radial direction. Assuming steady state and assuming that the disc is azimuthally symmetric, the only terms that do not vanish are:

$$\frac{v_\phi^2}{r} = \frac{GM_*}{r^2} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{82}$$

Substituting $P = c_s^2 \rho$, we obtain

$$\begin{aligned} \frac{v_\phi^2}{r} &= \frac{GM_*}{r^2} + \frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{\mathrm{d}c_s^2}{\mathrm{d}r} = \frac{GM_*}{r^2} + \frac{c_s^2}{r}\frac{\mathrm{d}\log\rho}{\mathrm{d}\log r} + \frac{c_s^2}{r}\frac{\mathrm{d}\log c_s^2}{\mathrm{d}\log r} = \\ &= \frac{GM_*}{r^2} - (p+2q)\frac{c_s^2}{r}. \end{aligned} \tag{83}$$

We can then write the velocity as

$$v_\phi = v_K \left[1 - \frac{c_s^2}{v_K^2}(p+2q)\right]^{1/2}, \tag{84}$$

where we have used the Keplerian velocity $v_K = \sqrt{GM_*/r}$.

(iv) To connect the formula we just derived to the requirement that the disc is thin we notice that $c_s/v_K = H/R$, which is a consequence of the fact that $H = c_s/\Omega_K$. The azimuthal velocity is thus

$$v_\phi = v_K \left[1 - \left(\frac{H}{r}\right)^2 (p+2q)\right]^{1/2}, \tag{85}$$

and the second term is small provided that $H \ll r$. In this case we can expand the expression to

$$v_\phi \simeq v_K \left[1 - \frac{1}{2}\left(\frac{H}{r}\right)^2 (p+2q)\right] \tag{86}$$

and the fractional difference from Keplerian rotation is $\frac{1}{2}\left(\frac{H}{r}\right)^2(p+2q) = \mathcal{O}((H/r)^2)$.

(v) The calculation proceeds largely as before, except for a) the gravity term, which now depends on the spherical distance $R^2 = r^2 + z^2$. We can expand this for the case $(z/r)^2 \ll 1$ and obtain

$$\frac{GM_*}{R^2} = \frac{GM_*}{r^2}\left(1 - \frac{3}{2}\frac{z^2}{r^2}\right) \tag{87}$$

b) the term $\frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}$. When computing the derivative, we have to keep in mind that $\rho = \rho_0 \exp(-z^2/2/H^2)$ and so this produces a new term:

$$\frac{c_s^2}{\exp\left(-\frac{z^2}{2H^2}\right)}\frac{\partial}{\partial r}\left[\exp\left(-\frac{z^2}{2H^2}\right)\right]. \tag{88}$$

While z does not depend on r, $H = c_s/\Omega$ does because both c_s and Ω do and so we must apply the chain rule. This produces two new terms, one of which cancels out with the one we derived in a).

After some algebra we get to the final result, which is:

$$v_\phi = v_{K,mid} \left[1 - \left(\frac{H}{r} \right)^2 \left(p + 2q + q\frac{z^2}{H^2} \right) \right]^{1/2}. \tag{89}$$

Note that we have defined $v_{K,mid} = \sqrt{\frac{GM_*}{r}}$, the value in the midplane. The new term is negative and so for $z \neq 0$ the material rotates slower than in the midplane.

9 Particle physics

Magnetars, namely a type of neutron stars with an extremely high magnetic field, are good candidates to be the sources of ultra-high energy cosmic rays, which are the most energetic particles ever observed. However, it's likely that they can accelerate up to the highest energies (~ 100 EeV) only in the very early stages of their life. One of the closest known Magnetars is SWIFT J1818.0-1607, which is estimated to be only 240 years old. It is located 4.4 kpc away from the Earth, on the galactic plane. Suppose SWIFT J1818.0-1607 emitted for a very small time at its birth cosmic rays, and that they consist only in three components: protons (P), neutrons (N) and iron nuclei (I), each following its own emission spectrum:

$$N_\mathrm{P} = N_{0_\mathrm{P}} \left(\frac{E}{E_0}\right)^{-2.5} \quad N_\mathrm{N} = N_{0_\mathrm{N}} \left(\frac{E}{E_0}\right)^{-2.5} \quad N_\mathrm{I} = N_{0_\mathrm{I}} \left(\frac{E}{E_0}\right)^{-2.2} \quad ,$$

where $E_0 = 10^{17}$ eV, $N_{0_\mathrm{P}} = 10^{40}$, $N_{0_\mathrm{N}} = 10^{38}$, $N_{0_\mathrm{I}} = 10^{39}$ and consider a hard cutoff at $E_C = 2 \cdot 10^{20}$ eV.

i) **(4 points)** Consider only particles with $E > E_0$. What would the spectrum of particles observed at Earth in 1783 be? How many events would be observed now by the Pierre Auger Observatory (area= 3000 km^2)? At which energy? Imagine that the Galactic Magnetic Field between us and the magnetar is uniform and constant, with a direction perpendicular to the galactic plane and its intensity is 1 μG.

Hint: consider $\arcsin x \approx x + \frac{x^3}{6}$.

ii) **(3 points)** Cosmic rays reach the Earth and immediately interact with the atmosphere. In the first interactions Kaons are often produced. Imagine that a K^+ with energy E=1 TeV is produced in the high atmosphere (h=10 km). Which particles would it most likely decay into? Compute the energies in the Earth's reference system of its daughter particles. Are they likely to reach the ground or will they decay again? In which particles? (consider only the most probable decays)

iii) **(3 points)** A muon produced in the high atmosphere (at an altitude h =1000 m) moves towards Earth in a radial direction, on the equatorial plane from the projection of the Greenwich meridian. Suppose that the Earth magnetic field has a value of 0.5 G, that it is constant and that it is oriented towards the North Pole (ignoring the discrepancy between the magnetic North Pole and the geographical one). What momentum is needed for the muon so that it hits the city of Nairobi (approximately longitude 37°E and latitude 0°)? Consider the Earth radius to be 6400 km. If the muon has the momentum you just computed, what is its probability to reach Nairobi, if the average life of a muon is $\tau_\mu = 2 \cdot 10^{-6}$ s?

Solutions

(i) Protons and Iron nuclei are charged particles and thus are deflected (and delayed) by magnetic fields. So, when the Magnetars emits cosmic rays, the only component that gets to the Earth together with the light is the neutron one. Neutrons, however, decay with a lifetime of roughly 15 minutes. Their decay time is boosted by the relativistic time delay so that their mean decay path (considering they travel at c) is:

$$\lambda = 9.2 \text{ kpc} \cdot E[\text{EeV}] = 0.92 \text{ kpc} \cdot \frac{E}{E_0}.$$

There is also a geometric factor to be computed:

$$\phi_0 = \frac{N_{0_\text{N}}}{4\pi D^2} = 4.2 \cdot 10^2 \text{ km}^{-2} \ ,$$

so the final flux will be:

$$\phi_{1783} = \phi_0 \left(\frac{E}{E_0}\right)^{-2.5} \cdot e^{-\frac{D}{\lambda}} \ ,$$

where D is the distance between the Magnetar and the Earth. Neutrons all arrive together at that time, so none will be arriving nowadays. However, since we consider a uniform and stable magnetic field without a random component, only the particles that have a specific curvature radius, so that they are delayed exactly 240 years, will arrive on Earth. This can be computed by measuring the arc connecting two dots separated by a distance D (see Fig. 7).

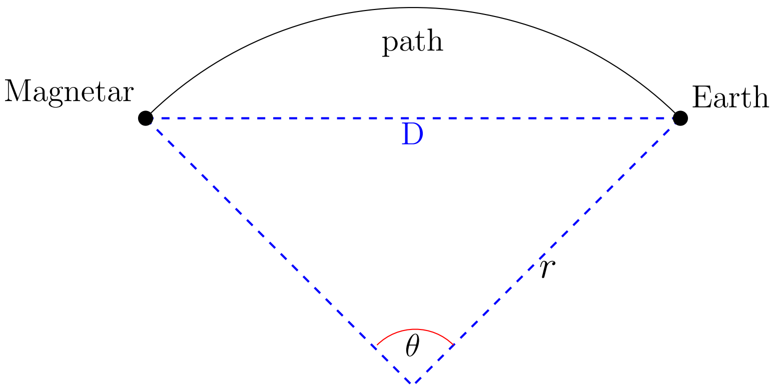


Figure 7

Thus, we have $L_\text{path} = r\theta$ with $\theta = 2 \arcsin \frac{D}{2r}$ which for simplicity we will expand as:

$$\theta \approx 2\left(\frac{D}{2r} + \frac{D^3}{48r^3}\right) \ .$$

Consequently, $L_\text{path} = D + \frac{D^3}{24r^2}$, from which we can extract $r = \left(\frac{D^3}{24(L_\text{path}-D)}\right)^{\frac{1}{2}}$, where $L_\text{path} - D = 240\text{ly}$. Then we can compute the radius of a particle in a magnetic field:

$$r(m) = \frac{E[\text{GeV}]}{0.3 \ q[\text{eV}] \ B[\text{T}]} \ ,$$

and from that, after the appropriate unit conversions we receive particles with energy $E_{\mathrm{P}} = 6.43 \cdot 10^{18}\mathrm{eV}$ for protons and $E_{\mathrm{I}} = 1.67 \cdot 10^{20}\mathrm{eV}$ for iron nuclei. Then, we can compute the flux, recalling that only one energy is arriving at this very moment on Earth:

$$\phi_{\mathrm{P}} = \frac{N_{0_{\mathrm{P}}}}{4\pi L^2}\left(\frac{E_{\mathrm{P}}}{E_0}\right)^{-2.5}$$

and

$$\phi_{\mathrm{I}} = \frac{N_{0_{\mathrm{I}}}}{4\pi L^2}\left(\frac{E_{\mathrm{I}}}{E_0}\right)^{-2.2} \quad .$$

In order to get the number of events observed by Auger, we can simply multiply the flux by the surface of Auger and get:

$$N_{\mathrm{P}}^{\mathrm{Auger}} = \phi_{\mathrm{P}} \cdot A_{\mathrm{Auger}} = 3780$$

for protons, and:

$$N_{\mathrm{I}}^{\mathrm{Auger}} = \phi_{\mathrm{I}} \cdot A_{\mathrm{Auger}} = 1$$

for iron nuclei.

(ii) Kaons can decay in leptons or pions. The most probably decay for the first case is in muons and muon neutrino (not electron for helicity reasons) and in the second case in two pions $\pi^+\pi^0$. In either case we can compute the energy of the daughter particles for in the kaon rest frame by using the 4-vectors:

$$p_K = (m_K, \vec{0}), \quad p_\mu = (E_\mu, \vec{p}), \quad p_\nu = (E_\nu, -\vec{p})$$

We can then write $p_K = p_\mu + p_\nu$ and, since the neutrino is massless it's easier to isolate the muon and take the squares $(p_K - p_\nu)^2 = p_\mu^2$ that gives us $m_K^2 + 0 - 2(m_K E_\nu) = m_\mu^2$ from which we can obtain

$$E_\nu = \frac{m_K^2 - m_\mu^2}{2m_K} = |\vec{p}_\nu| = |\vec{p}_\mu|$$

and consequently

$$E_\mu = \sqrt{E_\nu^2 + m_\mu^2}$$

This is in the rest frame of the kaon, thus we need to convert this to lab reference frame, for that we apply the Lorentz transformation

$$E = \gamma_K(E^* + \beta_K P^* cos\theta^*)$$

where P is the total momentum of the particle in the CM, theta the angle of emission of the daughter particle with respect to the direction of the K and $\gamma_K = E_K/m_K$ and $\beta_K = p_K/E_K$. The distribution is isotropic in the CM frame so we expect a flat spectrum of products between a certain E_{min} and E_{max}, which are respectively the two extreme cases when $\theta^* = 0°$ and $\theta^* = 180°$.

We get then that $E_\nu^{\mathrm{min}} \sim 0\,\mathrm{GeV}$ and $E_\nu^{\mathrm{max}} \sim 954\,\mathrm{GeV}$. Neutrinos don't decay so there's no need to compute this part.

For muons, instead $E_\mu^{\min} \sim 46\,\text{GeV}$ and $E_\mu^{\max} \sim 1\,\text{TeV}$. Muons can decay and their decay time in the rest frame is $2.2 \times 10^{-6}s$, from which we can compute that $P_{\text{survival}} = exp(-h/\gamma c\tau)$ which is about 97% for the minimum muon energy and grows higher for the other ones. So muons from this decay are most likely to reach the ground.

The case of the decay $K^+ \to \pi^+\pi^0$ is very similar. In the kinematics computation the only difference is that we have no zero-mass particles here, giving as a result

$$E_0^* = \frac{m_K^2 + m_0^2 - m_+^2}{2m_k}$$

with the 0 subscript denoting the neutral pion and + the charged one. The result for the charged pion are symmetric and this way, computing as before the Lorentz boost one gets that $E_0^{\max} = 913\,\text{GeV}$, $E_0^{\min} = 82\,\text{GeV}$, $E_+^{\max} = 918\,\text{GeV}$ and $E_+^{\min} = 85\,\text{GeV}$. The mass of π^+ and the mass of π^0 can also be approximated as equal.

The large difference, in this case, regards the probability of reaching the ground: The charged pion decays via weak interaction and has a lifetime of $\tau_+ = 2.5 \times 10^{-8}s$. while the neutral one decays via electromagnetic decay and has a lifetime of $\tau_0 = 8.5 \times 10^{-17}s$ making it for the second one impossible to reach the ground (its decay length is minuscule even if computed with the maximum possible energy: $\gamma_0^{\max} c\tau_0 = 1.72 \times 10^{-4}$ m . For the charged one, on the other side, the survival probability changes from 11% to 81% in the allowed energy range, meaning that we can expect part of the pions (mostly the most energetic) to reach the ground and part of them to decay (mostly the least energetic).

(iii) Let us choose a coordinate system centered at the center of the Earth, where the $\hat{x}$ axis represents the radial direction and the $\hat{y}$ axis is perpendicular to $\hat{x}$ and to $\hat{z}$, which is the North-South direction. Since the motion of the muon can be displayed on the x-y plane, we will not consider the $\hat{z}$ axis. The muon begins its life at a point $(x_0; 0)$ proceeding in the negative direction of the $\hat{x}$ axis (see Fig. 8).

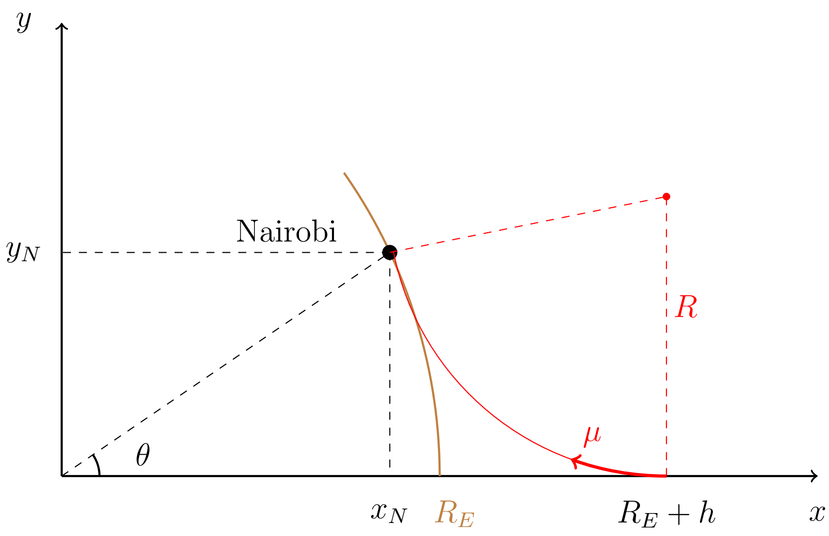


Figure 8

It will be deflected by the magnetic field with a radius of curvature

$$R[\mathrm{m}] = \frac{p[\mathrm{GeV}]}{0.3\ q[\mathrm{eV}]\ B[\mathrm{T}]}$$

towards the city of Nairobi $(x_N; y_N)$. It is possible to obtain the coordinates of Nairobi in our system:

$$(x_N; y_N) = (R_E \cos\theta; R_E \sin\theta)\ ,$$

where R_E is the radius of the Earth and θ is the longitude of Nairobi. It is possible to obtain the radius of curvature from the geometry of the system :

$$R = \frac{y_N^2 + (x_N - h - R_E)^2}{2y_N}\ ,$$

from which we can get the momentum of the muon

$$p[\mathrm{GeV}] = 0.3 \cdot q[\mathrm{eV}]\ B[\mathrm{T}]\ R[\mathrm{m}] = 32.1\mathrm{GeV}\ .$$

The survival probability depends on the length actually travelled by the muon, which is

$$L_{\text{eff}} = R \cdot 2 \cdot \arctan \frac{y_N}{R_E + h - x_N},$$

from which we can compute

$$P = \exp\left(-\frac{L_{\text{eff}}}{\gamma c \tau}\right) \approx 0.2\%\ ,$$

where $m_\mu = 105\ \mathrm{MeV}/c^2$, $\tau = 2.2\ \mu s$ and $\gamma = \frac{E}{m}$.

10 General relativity

Consider a small perturbation of the Minkowski spacetime produced by gravitational waves, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

i) **(0.5 points)** Show that, if n^μ is a null vector in Minkowski, then the vector $s^\mu = n^\mu - \frac{1}{2}\eta^{\mu\rho}h_{\rho\lambda}n^\lambda$ is null in the perturbed metric (always work to $\mathcal{O}(h)$).

ii) **(5 points)** Let us imagine s^μ is the trajectory of a photon emitted at initial frequency ν_0 by a source located at direction $\hat{n}$ from Earth. Compute, using the geodesic equation in the TT gauge, the change in the frequency of the photon, $(\nu(t) - \nu_0)/\nu_0$, produced by a monochromatic gravitational wave propagating along the z direction, $h_{\mu\nu} = h_{\mu\nu}(t - z)$.

iii) **(1 point)** Generalize the result for an arbitrary direction of propagation of the gravitational wave $\hat{\Omega}$.

iv) **(1 point)** What happens for $\hat{\Omega} \cdot \hat{n} = 1$ (*hint: answer this question assuming* $\hat{\Omega} = \hat{z}$)?

v) **(0.5 points)** Is the result linear, *i.e.* additive, for N gravitational waves propagating all in different directions?

vi) **(1 point)** Now suppose that gravitational waves have a modified dispersion relation $\omega(k) = c_s k$ ($c_s = 1$ in General Relativity). How would this new effect manifest in the gravitational redshift derived above?

vii) **(1 point)** In a real observation, gravitational waves are not perfectly chromatic. For example, in a binary system of two object with the same mass m, the frequency f of the gravitational waves evolves with time as

$$\frac{df}{dt} \sim 300 \left(\frac{Gm}{c^3}\right) f^{\frac{11}{3}} \tag{90}$$

during the in-spiraling phase. For $m = 10^{11}\mathrm{M}_\odot$ and $f \sim 1\mathrm{nHz}$, estimate how close to Earth must the source be in order to ignore the frequency evolution of the gravitational wave.

Solutions

Throughout this solution we use the (-,+,+,+) signature for the metric tensor.

(i) To $\mathcal{O}(h)$ we have:

$$
\begin{aligned}
g_{\mu\nu}s^\mu s^\nu =&(\eta_{\mu\nu}+h_{\mu\nu})(n^\mu - 1/2\eta^{\mu\rho}h_{\rho\lambda}n^\lambda)(n^\mu - 1/2\eta^{\nu\beta}h_{\beta\gamma}n^\gamma) = \\
=&\eta_{\mu\nu}n^\mu n^\nu + h_{\mu\nu}n^\mu n^\nu - 2\cdot\frac{1}{2}\eta_{\mu\nu}\eta^{\mu\rho}h_{\rho\lambda}n^\lambda n^\nu = 0
\end{aligned}
$$

(ii) and **(iii)** We now want to use the Geodesic Equation to study how the frequency of the emitted photon ν_0 changes as it reaches the Earth at time t, when our detector measures $\nu(t)$. In absence of gravitational waves we would measure ν_0 (we are neglecting any possible expansion of the Universe and other Doppler shifts). In the presence of a gravitational wave, the Geodesic equation for $s^0 = \nu$ reads:

$$
\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = -\Gamma^0_{\nu\rho}s^\nu s^\rho = -\Gamma^0_{\nu\rho}n^\nu n^\rho \ ,
$$

again to $\mathcal{O}(h)$. For the background vector n^μ we can write $n^\mu = \nu(1, -\hat{n})$, in the above equation, which implies, in the TT gauge:

$$
\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = -\frac{1}{2}\frac{\mathrm{d}h_{ij}}{\mathrm{d}t}\nu^2\hat{n}^i\hat{n}^j
$$

To proceed further we need to define the direction of propagation of the gravitational waves, Since we are asked to provide the solution for an arbitrary $\hat{\Omega}$, we will work out this case first, and then restrict ourselves to a wave propagating in the z-direction. We have

$$
\frac{\mathrm{d}h_{ij}(t-\hat{\Omega}\cdot\vec{x})}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\partial h_{ij}(t-\hat{\Omega}\cdot\vec{x})}{\partial t} + \frac{\mathrm{d}\hat{\Omega}\cdot\vec{x}}{\mathrm{d}\lambda}\frac{\partial h_{ij}(t-\hat{\Omega}\cdot\vec{x})}{\partial\hat{\Omega}\cdot\vec{x}} \ ,
$$

which can be further simplified by noticing that $\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \nu$ and that the metric perturbation is a function of $(t-\hat{\Omega}\cdot\vec{x})$, which allow us to arrive at

$$
\frac{\mathrm{d}h_{ij}(t-\hat{\Omega}\cdot\vec{x})}{\mathrm{d}\lambda} = \nu\frac{\partial h_{ij}}{\partial t}\left(1+\hat{\Omega}\cdot\vec{x}\right) \ ,
$$

where in the last equality we used $\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} = -\nu\hat{n}$ for the incoming photon. The geodesic equation finally becomes:

$$
\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = -\frac{1}{2}\frac{\nu}{(1+\hat{\Omega}\cdot\hat{n})}\frac{\partial h_{ij}}{\partial\lambda}\hat{n}^i\hat{n}^j \ .
$$

To $\mathcal{O}(h)$ we can integrate the above equation, then expand the solution to $\mathcal{O}(h)$ and finally obtain

$$
\frac{\nu(t)-\nu_0}{\nu_0} = \frac{\hat{n}^i\hat{n}^j}{2(1+\hat{\Omega}\cdot\hat{n})}\Delta h_{ij} \ ,
$$

where $\Delta h_{ij} \equiv h_{ij}(t_p - \hat{\Omega}\cdot\vec{x}_p) - h_{ij}(t_e - \hat{\Omega}\cdot\vec{x}_e)$ is the difference in the amplitude of the gravitational wave between the emission time t_p and the time it is received on Earth at t_e.

For a wave propagating in the z-direction we have $\hat{\Omega} = \hat{z}$, the metric perturbations can be written as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\hat{n} = -(\sin\theta\cos\phi)\hat{x} - (\sin\theta\sin\phi)\hat{y} - \cos\theta\hat{z} \ .$$

The change in frequency becomes

$$\frac{\nu(t)-\nu_0}{\nu_0} = \frac{\sin^2\theta}{2(1+\cos\theta)}[(\cos^2\phi - \sin^2\phi)\Delta h_+ + 2\sin\phi\cos\phi\Delta h_\times]$$

which is equal to 0 if $\cos\theta = 1$.

(iv) This remains true for $\hat{\Omega}\cdot\hat{n} = 1$, and it is due to the transverse nature of the gravitational waves.

(v) Notice that in the above derivation we never had to assume that all gravitational waves propagate in a single direction, *i.e.* the result is additive for multiple waves.

(vi) In the case of modified dispersion relation, $c_s \equiv \omega/|\vec{k}| \neq 1$, the equation of motion for the propagation of gravitational waves becomes

$$\left(-\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) h_{\mu\nu} = 0 \ ,$$

whose solutions are a function of $(t - \hat{\Omega}\cdot\vec{x}/c_s)$, such that all the result presented above apply by sending $\hat{\Omega} \to \hat{\Omega}/c_s$.

(vii) The typical time-scale over the signal evolves for the system presented in the text is of the order

$$\Delta t = \frac{f}{\mathrm{d}f/\mathrm{d}t} \sim 1.1\mathrm{kpc/c}.$$

In order to obtain the distance D

$$D = c\Delta t =$$

which gives us a distance of $\mathcal{O}(\mathrm{kpc})$.