Contents

1. A gentle persuasion

Prof. dr. Wim Beenakker
Radboud University
11 points

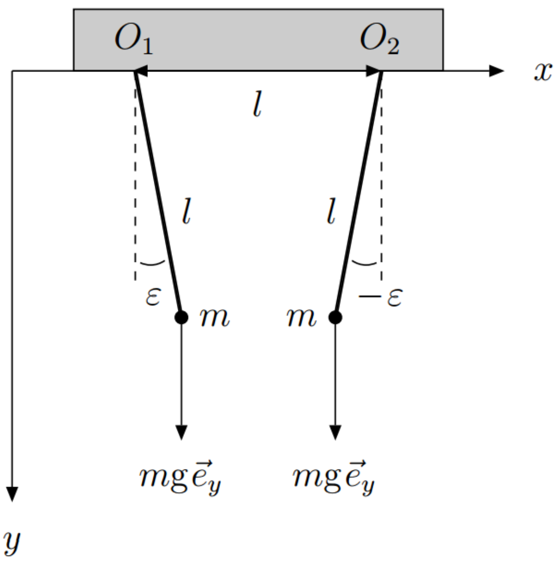

Figure 1.1

Consider a system comprised of two pendulums suspended from fixed, frictionless supports at positions $O_1 = (X_1, 0)$ and $O_2 = (X_1 + l, 0)$. Each pendulum consists of a rigid massless wire with constant length l at the end of which a point particle with constant mass m is attached. The motion of the system takes place in the xy-plane as a result of a net constant gravitational field with acceleration g in the positive y-direction. In addition to this net external gravitational field we also take into account the very weak gravitational attraction between the two point particles. In the figure a sketch is given of the equilibrium position of the aforementioned system, with an exaggerated representation of the angular displacement ε from the vertical position.

Use the following polar coordinates to describe the relative positions of the point particles with respect to the corresponding fixed suspension points:

$$(x_1, y_1) = (X_1 + l\sin\varphi_1\ ,\ l\cos\varphi_1)$$
$$(x_2, y_2) = (X_1 + l + l\sin\varphi_2\ ,\ l\cos\varphi_2)$$

with $\varphi_{1,2}$ denoting the angular displacements with respect to the vertical positions of the two pendulums.

1.a [3 p]

- Show that for small oscillations of the pendulums and a vanishing external gravitational energy at $y = 0$, the Lagrangian of the system can be approximated by

$$L \approx \frac{1}{2} m l \Big[l(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + (\mathrm{g}' - \mathrm{g})(\varphi_1^2 + \varphi_2^2) - 2\mathrm{g}'\varphi_1\varphi_2 + \mathrm{g}'\varphi_1 - \mathrm{g}'\varphi_2 + C \Big]$$

  with $\dot{\varphi}_{1,2}$ denoting the time derivative of $\varphi_{1,2}$.

- Express the constants C and g' in terms of Newton's constant G_N.

1.b [1 p] Derive the equations of motion for $\varphi_{1,2}$.

1.c [1 p] Determine the equilibrium angle ε in terms of g and g'.

1.d [3 p] Find the eigenmodes of the system.

1.e [3 p]

- Solve the equation of motion if at initial time $t = 0$

$$\varphi_1(0) = \varepsilon + \varphi_0$$
$$\varphi_2(0) = -\varepsilon$$
$$\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0$$

  which for $\mathrm{g}' \ll \mathrm{g}$ implies that initially only the first pendulum is swinging.

- How long will it take until the situation is completely reversed and only the second pendulum is swinging? Plug in the following numerical input:
  $m = 1$ kg, $l = 1$ m, $\mathrm{g} = 9.81$ m s^{-2} and $G_\mathrm{N} = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

2. Optical Cavities: The Fabry-Perot Etalon

Dr. Nandini Bhattacharya
TU Delft
8 points

An interferometer which can be found quite often inside a laser is the Fabry-Perot interferometer. It is named after Charles Fabry and Alfred Perot who developed the Fabry-Perot etalon (FPE) in 1899. This device mainly consists of two reflecting or partially reflecting surfaces. It is used in many applications which involve the control, storage and wavelength measurement of light.

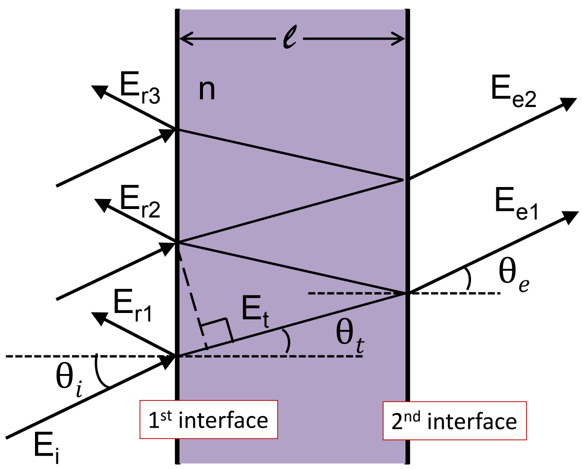

Figure 2.1 Fabry-Perot etalon showing the reflected and transmitted beams

The surfaces of a FPE are partly silvered and must be very smooth and planar. Also their relative tilt has to be very small. In Figure 2.1 we can see such an etalon. The optical path length l_{opt} can be determined to be $l_{opt} = nl$, if the distance l between the reflecting surfaces and the refractive index n is precisely known. Light of wavelength λ with an amplitude of E_i and wave vector $k = \frac{2\pi}{\lambda}$ is incident on the etalon at an angle θ_i.

2.a [0.5 p] How is the angle of emergence θ_e related to the angle of incidence θ_i for a parallel plate?

2.b [1 p] Determine the phase difference δ between two adjacent beams which leads to constructive interference.

2.c [1 p] The FPE consists of two interfaces and we must consider reflection and transmission coefficients at both separately. For the first interface we can write $r = \frac{E_r}{E_i}$ and $t = \frac{E_t}{E_i}$ as the reflection and transmission coefficients respectively as in Figure 2.2. Here E_r represents the reflected amplitude and E_t the transmitted amplitude. Similarly for the second interface we have r' and t' as the reflection and transmission coefficients. We assume $n > 1$.

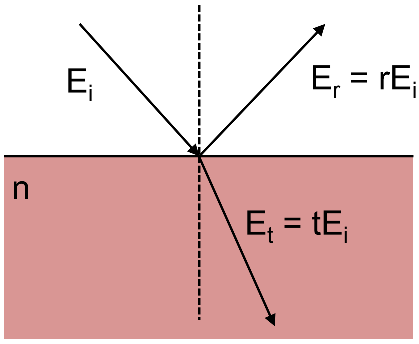

Figure 2.2 Fabry-Perot etalon showing the reflected and transmitted beams

- Sketch a figure reversing the direction of rays in Figure 2.2 where the amplitudes are written in terms of the amplitudes reflectivities and transmissivities to build the incident ray.
- Combine your sketch above with Figure 2.2 to make a new sketch where all the amplitudes are expressed in terms of the incident amplitude E_i. Consider conservation of energy for this figure and all possible rays thus also a reflected ray in the medium.

2.d [1 p] Use the figures above to derive the *Stokes relations* between the amplitude reflectivities and transmissivities.

$$tt' = 1 - r^2$$

$$r = -r'$$

From the *Stokes relations* above what can you conclude about the reflection at the second media?

2.e [1.5 p] We will now add the individual amplitude contributions of the reflected partial waves and transmitted partial waves to calculate the total reflection and transmission amplitudes. Using $e^{i\delta}$ as the total accumulated phase shift after one round trip, where phase jumps due to reflection have been included we can sum up the transmitted partial waves and reflected partial waves contributing to the field amplitude on either side of the etalon.

Using Figure 2.1 and what you have obtained in the sections above calculate $E_{tr} = \sum_{m=1}^{\infty} E_{\mathrm{e}m}$.

2.f [1 p] Using the expression derived above show that the transmitted intensity.

$$I_{tr} = I_{in} \frac{TT'}{1 - \sqrt{RR'} e^{i\delta}}$$

Here $r = \sqrt{R}$, $t = \sqrt{T}$ and $I_{in} = |E_{in}|^2$

2.g [0.5 p] What is the relation between R and T for a medium where there is no loss?

2.h [1 p] What do the interference fringes look like?

2.i [0.5 p] How would the fringes change if there were two wavelengths of light λ_1 and λ_2 incident on the etalon at the same angle of incidence?

3. Long range molecular interactions

Arthur Christianen
PION winner 2016
11 points

To describe the properties of all states of matter, it is important to know how molecules interact with each other. Not only on a short range, but also when the distance between the molecules is bigger than the molecular size. For low energy molecular collisions for example, it is often enough to know about the long range potential to describe the collision process. When the molecules don't come very close to each other it is often sufficient to only consider electrostatic interactions.

3.a [1 p] One of the most important interactions between molecules or atoms on a short range, is the exchange interaction, or Pauli repulsion. This interaction is caused by the need for fermionic wave functions to be antisymmetric. Why don't we have to consider this interaction when the intermolecular separation is large?

In this exercise we consider closed-shell molecules so that there are no spin-spin interactions.

To determine the form of the long range electrostatic potential, we first consider a very classical picture. Consider two molecules (A and B), each consisting of multiple point charges q_i and q_j with coordinates $\mathbf{r_i}$ and $\mathbf{r_j}$ respective to the centers of mass of the molecules. i and j are here the indices that label the atoms of molecule A and B respectively. The distance between the centers of mass is given by $\mathbf{R}$. Because we are interested in the long range potential we can consider $\mathbf{R}$ to be much larger than the intermolecular coordinates $\mathbf{r_i}$ and $\mathbf{r_j}$. In this whole exercise we will use atomic units, where $\frac{1}{4\pi\epsilon_0} = 1$

3.b [3 p] Neglect terms with powers of $\frac{1}{\mathbf{R}}$ higher than 3, and take $\mathbf{R}$ to be along the z-axis. Show that the resulting electrostatic potential energy, is given by:

$$V(R) = \frac{q_A q_B}{R} + \frac{q_B \mu_{A,z} - q_A \mu_{B,z}}{R^2} + \frac{2q_A Q_{B,zz} + 2q_B Q_{A,zz} + \mu_A \mathbf{T} \boldsymbol{\mu_B}}{R^3} \tag{3.1}$$

Here, q_A and q_B are the total charges of the molecules, μ_A and μ_B are the dipole moments of the molecules. $\mathbf{T}$ is a second rank tensor given by:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$Q_{A,zz}$ and $Q_{B,zz}$ are the zz-components of the quadrupole moment tensors given by:

$$Q_{X,zz} = \sum_k q_k(3z_k^2 - |\mathbf{r_k}|^\mathbf{2})$$

To apply this expression quantum mechanically, time independent perturbation theory is used. The electrostatic potential is considered as a perturbation on the Hamiltonians of the individual, non-interacting molecules. Because the molecules don't interact in the unperturbed case, the total unperturbed wave function can be written as the product of the two molecular wave functions:

$$\Psi^{(0)} = \psi_A^{(0)}\psi_B^{(0)}$$

The potential of equation 3.1 is turned into the perturbing Hamiltonian by simply replacing the expressions for the dipole and quadrupole moments by the corresponding operators. For the following questions, assume the molecules are in their ground state. Assume all the properties of the individual molecules are known.

3.c [2 p] Now consider only a charge-dipole interaction term:

$$\hat{V} = \frac{q_B \hat{\mu}_{A,z}}{R^2}$$

Calculate the energy corrections for this term in first and second order perturbation theory. Expressions for these energy corrections are given at the end of the exercise. Show that second order perturbation theory gives an energy correction of the form:

$$E_0^{(2)} = -\frac{q_B^2}{2R^4}\alpha_{A,zz}$$

Determine an expression for $\alpha_{A,zz}$. What is the physical meaning of this variable?

3.d [3 p] Now consider the dipole-dipole interaction term:

$$\hat{V} = \frac{\hat{\mu}_A \mathbf{T} \hat{\boldsymbol{\mu}}_\mathbf{B}}{R^3}$$

Again calculate the first and second order energy corrections. Show that the second order energy correction can be separated into three terms: two terms where the molecular properties of A and B are separated and one term where they are mixed. Explain for each of these three terms if they have a classical equivalent or if they are purely quantum mechanical.

3.e [2 p] What are the leading long-range electrostatic interaction terms for the following pairs of molecules? With which order of R do the interactions fall of?

- H^+ and He
- He and He

- Benzene and benzene.
  Benzene is a six-membered flat carbon ring, with a 6-fold symmetry.
- HCl and HCl

Perturbation theory and Dirac notation

Given is a Hamiltonian, which consists of a large contribution $\hat{H}_0$ of which the solutions are known (the unperturbed Hamiltonian) and a small perturbation $\hat{H}'$:

$$\hat{H} = \hat{H}_0 + \hat{H}'$$

The first and second order energy corrections on the unperturbed energies following from perturbation theory are given by:

$$E_n^{(1)} = \langle \psi_n^0 | \hat{H}' | \psi_n^{(0)} \rangle$$

$$E_n^{(2)} = \sum_{n \neq m} \frac{|\langle \psi_n^{(0)} | \hat{H}' | \psi_m^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Here the wave functions and energies marked with (0) are the solutions of the unperturbed Hamiltonian. The n and the m label the solutions of the unperturbed Hamiltonian.

The notation with the brackets $\langle\rangle$ is called Dirac notation and means the following:

$$\langle \psi_1 | \hat{A} | \psi_2 \rangle = \int \psi_1^*(\tau) \hat{A} \psi_2(\tau) d\tau$$

Where the integral is over all space and τ are all relevant coordinates for the wave function.

4. Cesium hyperfine structure and atomic clocks

Prof. dr. Steven Hoekstra and Kevin Esajas, MSc
University of Groningen
12 points

Since 1967 the international standard of a second is defined as 9192631770 oscillation periods of the hyperfine frequency in the ground state of Cs-133. The Cs-133 atom has 55 electrons, with a single valence electron outside closed shells. This electron has spin s=1/2 while the rest of the electrons have a total spin of 0 in a symmetric antiparallel configuration. The nucleus of this atom has a nuclear spin I=7/2.

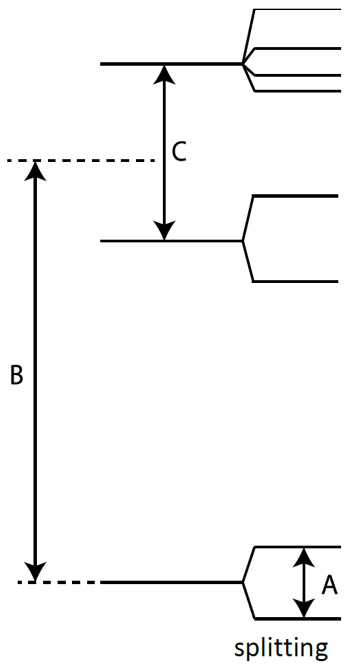

Figure 4.1 Several energy levels of the Cs-133 atom

4.a [2 p] In Figure 4.1 several energy levels of the Cs-133 atom are schematically drawn. The bottom energy level describes the ground state. The energy separation (splitting) between three sets of levels is denoted by A, B, and C. Complete the following table, by indicating

- the physical mechanism giving rise to the energy splitting
- the order of magnitude (by giving n, as in 10^n Hz) of the energy splitting

Energy splitting	Physical mechanism giving rise to splitting	Energy splitting size, order of magnitude n (in 10^n Hz)
A		
B		
C		

Table 4.1 Complete this table with the physical mechanism behind the energy splittings of Figure 4.1.

4.b [3 p] In the electronic ground state of Cs-133, the angular moment due to the electron spin and the nuclear spin couple together, resulting in hyperfine structure. When exposed to a weak magnetic field, it is the magnetic moment of the coupled nuclear and electron spin that determines the magnitude of the shift of the hyperfine energy sublevel. The size of this shift is given by:

$$\Delta E|F, m_F\rangle = \mu_B g_F m_F B_z$$

where g_F is the hyperfine Landé g-factor, μ_B the Bohr magneton, m_F the magnetic projection quantum number, and B_z an external magnetic field. The g_F factor depends on the relative orientation of the angular moment contributing to the total angular momentum, and can in general be calculated by evaluating the following two formulas:

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

$$g_F = \frac{F(F+1) - I(I+1) + J(J+1)}{2F(F+1)} g_J$$

- Evaluate the magnitude of the Zeeman splitting in a weak magnetic field for the two hyperfine levels of the electronic ground state, and express these in units of $\mu_B B_z$.
- The hyperfine energy levels, which are degenerate when there is no external field, split into several sublevels in a magnetic field. For a weak and strong field the grouping of these energy levels is different, since in strong magnetic fields the nuclear spin is decoupled from the electron spin. Complete the table below.

	Weak magnetic field	Strong magnetic field
Higher group of energy levels	# levels = ...	# levels = ...
Lower group of energy levels	# levels = ...	# levels = ...

Table 4.2 Complete this table with the number of sublevels in two groups of energy levels, both in weak and strong magnetic B fields.

4.c [3 p] Consider an atomic beam of Cs-133 in the ground state which is sent on the y-axis through an interaction region with magnets. A cut-through (X-Z plane) of 5 possible magnet arrangements is shown below in Figure 4.2. Assume that the dimensions of the beam are as indicated by the grey circle, and that the beam before the experiment is nicely collimated, i.e. the transverse velocity is zero throughout the beam. You can also assume that the atoms are in the strong-field regime during the interaction with the magnetic field. The beam propagates into the paper at some velocity v. Make a sketch of the trajectory of the beam in the y-z plane, showing the effect of the magnetic field on the atoms:

- if they are all in the lower hyperfine level
- if they are all in the $m_F = +4$ state, and that the atoms enter the local magnetic field adiabatically.

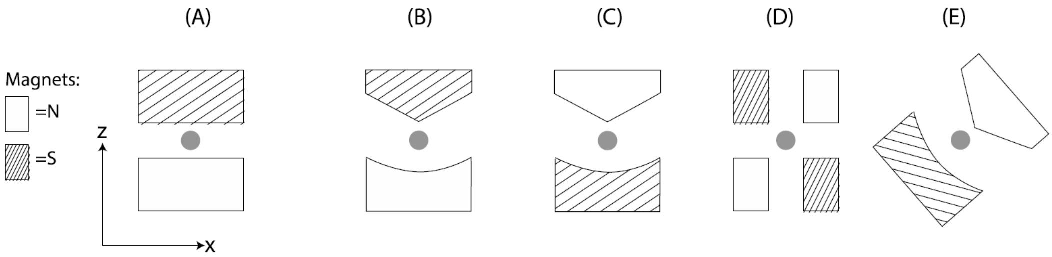

Figure 4.2 Various magnet arrangements, through which an atomic beam of Cs propagates (grey dot).

4.d [4 p] To make use of the hyperfine splitting of the cesium atom as a clock a setup as show in Figure 4.3 can be used. This setup consists of an oven producing a beam of Cs-133 atoms in the electronic ground state. The atomic beam passes through a strong and inhomogeneous magnetic field, created by a set of permanent magnets of opposite polarity and shape as shown, labelled 'A Magnet'. Part of the beam then passes through a microwave field, tuned to drive the field-free transition between the two hyperfine states. This is done in a weak uniform magnetic background 'C field'. Then the beam passes again through a set of strong permanent magnets, labelled 'B Magnet'. Finally the atoms are ionized and collected, resulting in the detector signal.
Complete the table below, indicating the (F,m_F) states in which the atoms are at the numbered locations.

	Location (1)	(2)	(3)	(4)
(F, m_F)	...	...	...	...

Table 4.3 Complete the table, indicating the hyperfine components present at the indicated points (1)-(4) in Figure 4.3.

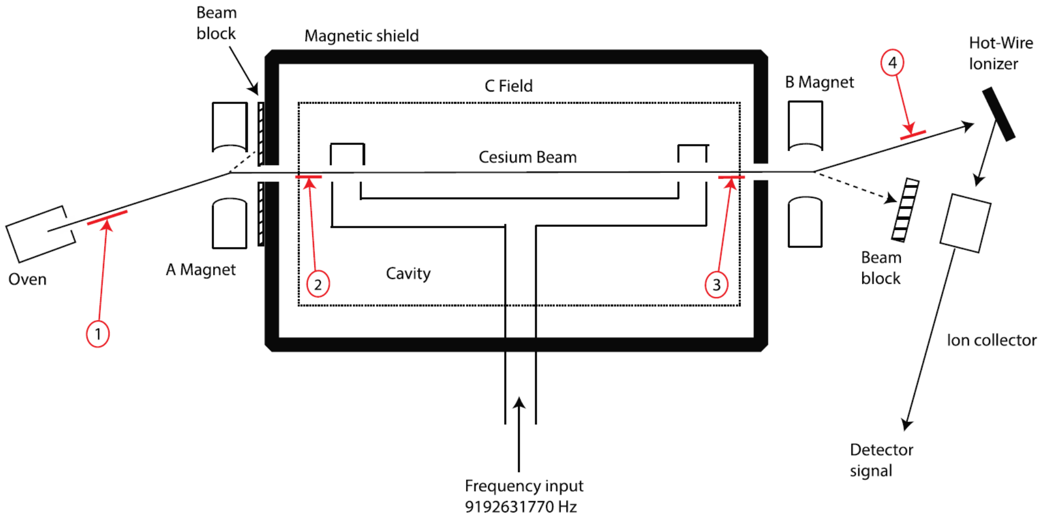

Figure 4.3 An atomic clock setup based on an atomic beam of Cs-133 atoms.

5. Some beer physics

Dr. Arie van Houselt
University of Twente
10 points

When there is liquid between a glass and a beer mat, the capillary force can make the mat stick to the bottom of the glass. See the sketch below (not to scale).

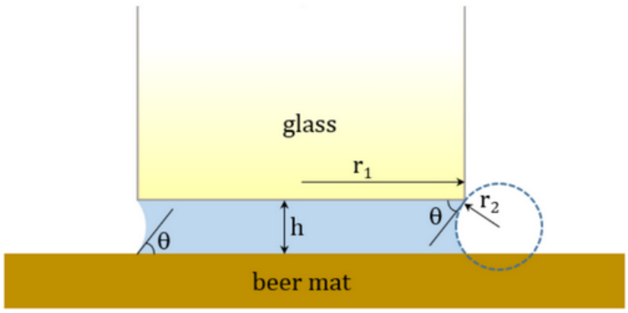

Figure 5.1

5.a [3 p] Show that the pressure difference(ΔP) across the liquid gas interface is given by the Laplace equation:

$$\Delta P = \gamma(\frac{1}{r'} + \frac{1}{r''})$$

Here γ is the surface tension and r' and r'' are the radii of curvature at the interface.

The following data is given:
Glass diameter d = 2 r_1 = 6 cm;
$m_{beermat} = 7$ g;
$\theta = 80^\circ$;
Surface tension γ for 5 % ethanol solution in water = 70 mN m^{-1}.

Assume in the question below that the liquid between glass and beer mat is a 5% ethanol solution in water and that the Laplace pressure balances the gravitational force.

5.b [2 p] Calculate the maximum thickness h at which the liquid film is still able to stick to the beer mat.

Beer contains, besides water and alcohol, different proteins, which play a role in the foam formation.

5.c [1 p] Explain whether the maximum film thickness will increase or stay constant or decrease when you take the presence of these proteins into account.

The decomposition of ethanol in the human body after consumption of beer is usually treated as zero order process with a constant decomposition rate. The decomposition of ethanol is actually a two-step process, consisting of an adsorption step, followed by oxidation of ethanol by the enzyme LADH (Liver Alcohol Dehydrogenase). Mathematically these steps can described by the following consecutive reaction sequence:

$$A \xrightarrow{k_1} B \tag{5.1}$$

$$B \xrightarrow{k_2} C \tag{5.2}$$

The first step (adsorption) exhibits first order kinetics: $\frac{dA}{dt} = -k_1 A$. The second step (oxidation) exhibits zero order kinetics with a constant rate k_2. Directly after consumption at t = 0, only A is present (with amount A_0).

5.d [3 p] Find an expression for the amount of B as a function of time. At what time t_m does B exhibit a maximum? What is the amount of A at time t_m?

In the plot below the amounts of A, B and C in the human body are shown after consumption of 1 mole alcohole (which corresponds roughly to one serving of an alcoholic beverage) using the typical values $k_1 = 2.89 \times 10^{-3} s^{-1}$ and $k_2 = 4.44 \times 10^{-5}$ mol s^{-1}

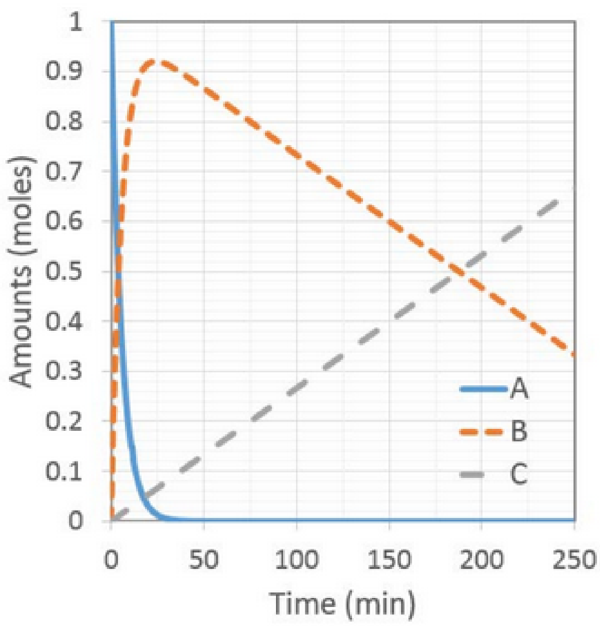

Figure 5.2

5.e [1 p] Explain that the zero order treatment is justified.

6. Dark Matter and Gravitational Lensing

Dr. Søren Larsen
Radboud University
10 points

The *visible* component of spiral galaxies, such as our own Milky Way, is dominated by a flattened, rotating disc. In a typical spiral galaxy, this disc contains most of the stars and gas. In this exercise, we assume that objects belonging to the disc follow circular orbits around the centre of the galaxy, and that these orbits are all confined to a single plane.

Measurements of the orbital velocities as a function of distance from the centre, $v(R)$, yield the *rotation curve*. Outside of the very central regions of galaxies, rotation curves are typically flat (see figure below). The flat rotation curves extend well beyond the regions containing most of the visible matter, indicating that most of the mass is invisible. In this exercise we assume that most of the *mass* belongs to a spherically symmetric *dark halo*.

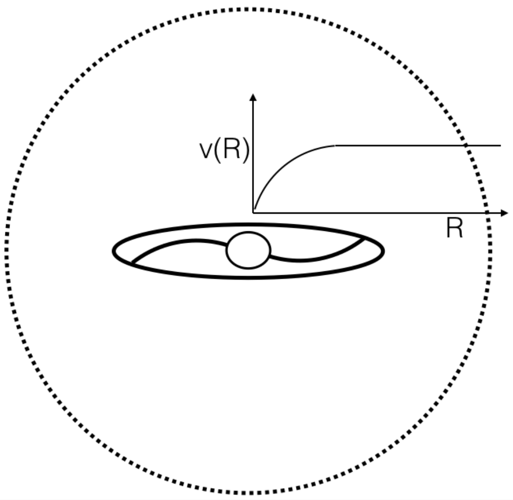

Figure 6.1

Let v_c be the rotational velocity on the flat part of the rotation curve.

6.a [2 p] Find expressions for the total mass $M(R)$ and density $\rho(R)$ of the dark halo in terms of v_c and the radius, R.

The nature of dark halos remains one of the great unsolved problems of modern cosmology. One suggestion is that they might consist of relatively massive 'compact objects'(for example black holes), so-called 'MACHOs'(MAssive Compact Halo Objects). Such MACHOs might be detectable via the effect of *gravitational lensing* if they happen to pass in front of a background star, whose light would then be amplified.

In order to correctly calculate the deflection of a light ray in a gravitational field, one must use General Relativity. However, by treating photons as ballistic particles moving at the speed of light, a deflection is predicted even by classical mechanics.

Consider the geometry in the figure below: A photon is emitted by the star S, then passes within a distance $r_{\min}$ of the lensing mass at L, and is detected by an observer at O. As it passes near L, the photon is accelerated towards L and acquires a velocity $v_\perp$ perpendicular to the original direction of propagation. It is thus deflected by an angle α. We make the following assumptions:

- The parallel component of the velocity ($v_\parallel$) always differs negligibly from the speed of light, c.
- $v_\perp \ll c$ so that the angle α is small. For purposes of calculating the acceleration, it can then be assumed that the photon continues to move in a straight line (i.e., following the dashed line in the figure)
- The distances D_{SL} and D_{LO} are much greater than $r_{\min}$.

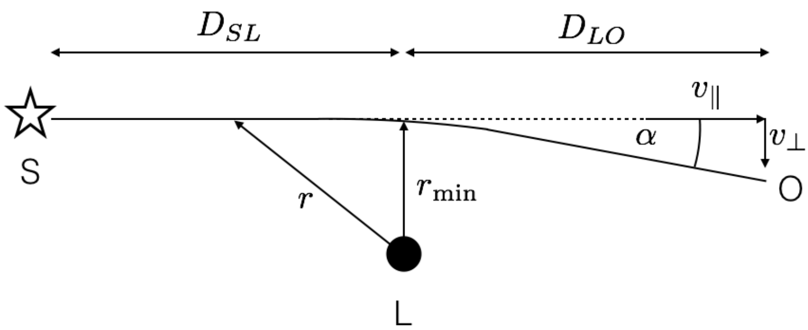

Figure 6.2

6.b [3 p] Use the laws of classical mechanics to find the 'Newtonian'deflection angle α as a function of the lensing mass M_L and the minimum distance $r_{\min}$.
Hint: Set time $t = 0$ at the moment the photon is closest to L, and integrate the acceleration over all times from $t = -\infty$ to $t = \infty$ to find $v_\perp$ at O.
You may find the following integral useful:

$$\int \frac{1}{(a+bx^2)^{3/2}}\mathrm{d}x = \frac{x}{a\sqrt{a+bx^2}}$$

According to the full general relativistic calculation (based on the *Schwarzschild solution* to Einstein's field equations), the deflection angle is

$$\alpha_{\mathrm{GR}} = \frac{4\mathrm{GM_L}}{\mathrm{r_{min}c^2}} \tag{6.1}$$

If the star, the lensing object, and the observer are exactly aligned with each other (i.e., all three are located along a single straight line) then the deflected light from the star will form an *Einstein ring*. Assume that the star is much farther away than the lensing object; $D_{SL} \gg D_{LO}$ and that the deflection occurs instantaneously as the light ray passes L.

6.c [2 p] Show that, with these assumptions (and using Equation 6.1), the radius of the Einstein ring, θ_E, is

$$\theta_E = 2\sqrt{\frac{GM_L}{D_{LO}c^2}}$$

We are now ready to return to the possible effect of gravitational lensing by MACHOs in the halo of the Milky Way.

Imagine drawing a ring with radius θ_E around each MACHO. If the line-of-sight from the observer towards a background star falls within θ_E, then the light from the star will be amplified by a factor of at least 1.34. The total fraction of the sky falling within the Einstein rings of all MACHOs is the *optical depth* to gravitational lensing, τ_L. (An equivalent definition of τ_L is that it gives the probability that a randomly selected star happens to amplified by at least a factor of 1.34.)

From the rotation curve, the total mass of the Milky Way dark halo is estimated to be about $4 \times 10^{11} M_\odot$, where $M_\odot$ is the mass of the Sun. Let us assume that all this mass is in the form of MACHOs located at a distance of 10^4 pc from the Earth. You can assume that the probability of two MACHOs having overlapping Einstein rings is negligibly small.

6.d [2 p] Show that the optical depth is independent of the masses of the individual MACHOs, and calculate the optical depth to gravitational lensing for a total dark halo mass of $4 \times 10^{11} M_\odot$.

In order to detect MACHOs, one looks for the *change* in brightness of a star as the MACHO moves in front of it. Suppose, as above, that we are looking for MACHOs at a distance of 10^4 pc. Assume also that the MACHOs are moving at velocities of 200 km/s perpendicular to the line-of-sight, and that the background stars (and the Earth) are stationary.

6.e [1 p] How long would it take for a MACHO with a mass of $1\,M_\odot$ to move an angular distance on the sky corresponding to the diameter of its own Einstein ring?

Constants:

1 pc $= 3.08 \times 10^{16}$ m
Speed of light: $c = 3 \times 10^8$ m s^{-1}
Gravitational constant: G $= 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
1 M$_\odot$ $= 1.989 \times 10^{30}$ kg

7. Superelectrodynamics

Dr. Alix McCollam
Radboud University
10 points

Superconductors have the property of zero electrical resistance, so that electrical current can flow without dissipation.

Consider a thin ring of material that becomes superconducting below a critical temperature T_c. This ring is placed in a weak external magnetic field at a temperature higher than T_c, as sketched in Figure 7.1a below.

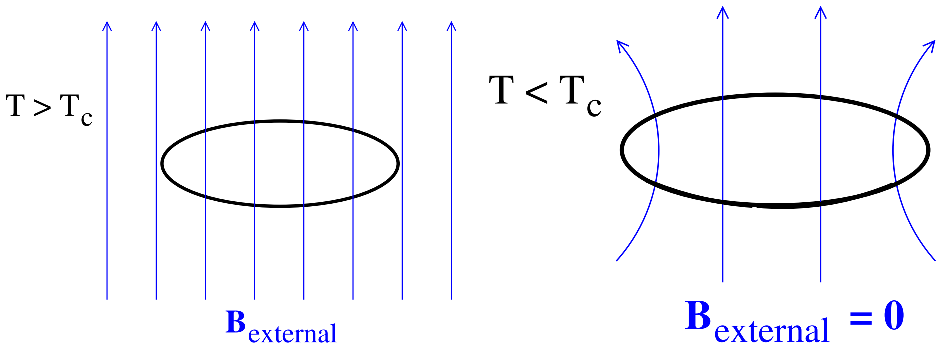

Figure 7.1

If the temperature is reduced so that $T < T_c$, and the external magnetic field is then switched off and completely removed, you will find that the magnetic field that threaded the hole in the ring is still there, as sketched in Figure 7.1b.

One way to understand this phenomenon is to model the ring as an RL circuit, as shown in Figure 7.2 (the magnetic field points out of the page in this sketch).

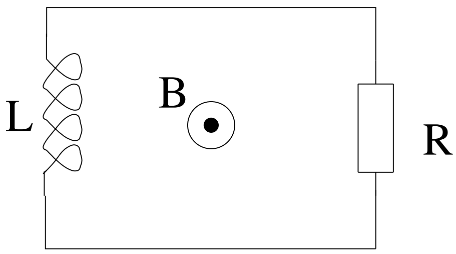

Figure 7.2

7.a [2 p] Show that $LI + BA =$ constant when the ring is superconducting, where L is the inductance of the ring, I is the current flowing in the ring, B is the strength of the external magnetic field and A is the area of the hole in the ring. Justify the steps in your answer, and explain what $LI + BA$ represents.

Superconductors are more than just a perfect conductors with $R = 0$. They also have the property that a bulk superconductor will expel magnetic field from its interior. In fact, the simplest type of superconductor will totally expel all magnetic field. This is called the Meissner effect and it distinguishes a superconductor from a perfect normal conductor. Consider the following (thought) experiment, illustrated in Figure 7.3.

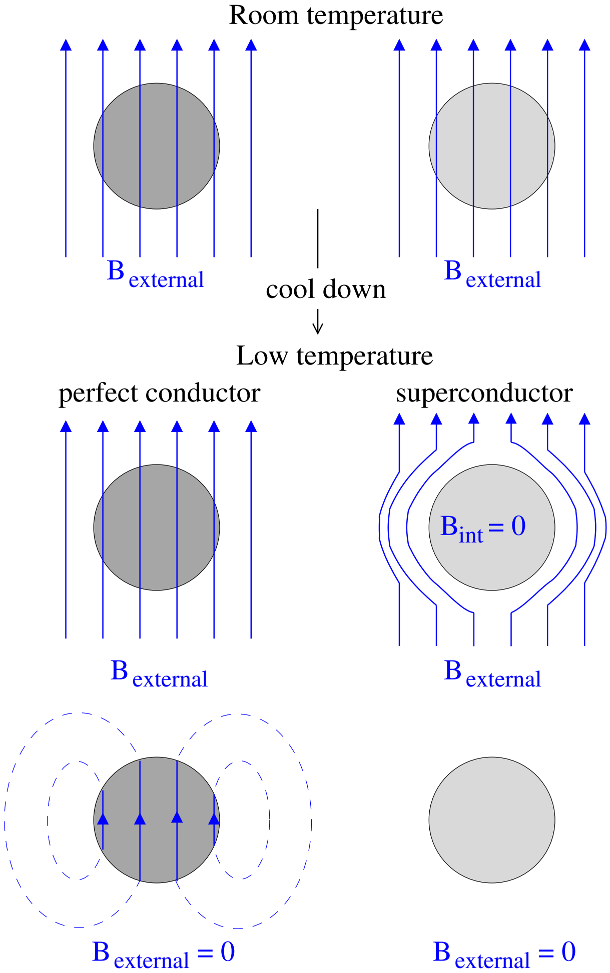

Figure 7.3

Solid spheres of two different materials are normal conductors with finite resistance at room temperature. Both spheres are placed in a weak external magnetic field and then cooled. At low temperature, one of the spheres becomes a perfect normal conductor with zero electrical resistance, and the other becomes a superconductor. The superconductor completely expels the magnetic field so that $B = 0$ in its interior. The magnetic field continues to pass through the perfect conductor unchanged. If the external magnetic field is then removed, the field *inside* the perfect conductor persists with the same strength and direction. The field inside the superconductor remains at zero.

The behaviour of a perfect normal conductor described above emerges naturally from the application of Maxwell's equations to the system. A conduction electron moving through a normal conductor under the influence of an electric field **E** experiences a force

$$\mathbf{F} = m_e \frac{\partial \mathbf{v}}{\partial t} = -e\mathbf{E} - \frac{m_e}{\tau}\mathbf{v} \tag{7.1}$$

here e, m_e and **v** are, respectively, the charge, mass and velocity of the electron, and τ is a scattering time related to electronic collisions that lead to the resistance of the material. In a perfect conductor, the electrons suffer no collisions, so $\tau \to \infty$.

7.b [1.5 p] If the current density in a normal conductor is

$$\mathbf{J} = -ne\mathbf{v} \tag{7.2}$$

where n is the number density of electrons, derive the relationship between **E** and **J** in a perfect conductor, and then apply the relevant Maxwell equations to show that

$$\nabla^2 \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\lambda^2}\frac{\partial \mathbf{B}}{\partial t} \tag{7.3}$$

where $\lambda = \sqrt{\dfrac{m_e}{\mu_0 n e^2}}$. Justify any assumptions you make.

The difference between the behaviour in magnetic field of a perfect conductor and a superconductor comes from a difference in the properties of a normal current and a supercurrent. Maxwell's equations apply in superconductors just as they do in normal conductors, but there is an additional constraint on the supercurrent density, namely

$$\boldsymbol{\nabla} \times \frac{m^*}{n^* q^2}\mathbf{J}_s = -\mathbf{B} \tag{7.4}$$

where $\mathbf{J}_s$ is the supercurrent density, and m^*, q and n^* are, respectively, the mass, charge and number density of śuperelectronsalso known as superconducting quasiparticles). Equation 7.4 is the London equation, and it leads to the Meissner effect.

7.c [1.5 p] Show that for a superconductor

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B} \tag{7.5}$$

where $\lambda_L = \sqrt{\dfrac{m^*}{\mu_0 n^* q^2}}$.

7.d [2 p] Consider a simple geometry where the perfect conductor or superconductor has a plane surface at $x = 0$ and the external magnetic field is in the y-direction, so that Equation 7.3 becomes

$$\frac{d^2}{dx^2}\left(\frac{\partial \mathbf{B}_y}{\partial t}\right) = \frac{1}{\lambda^2}\left(\frac{\partial \mathbf{B}_y}{\partial t}\right) \tag{7.6}$$

and Equation 7.5 becomes

$$\frac{d^2 B_y}{dx^2} = \frac{1}{\lambda_L^2} B_y \; . \tag{7.7}$$

Solve or write down solutions to the differential equations 7.6 and 7.7, and explain the physical significance of the terms λ and λ_L.

If a perfect conductor existed, it might have an electron density of $n \sim 10^{29}$ m^{-3}. The charge carriers in a superconductor are understood to be formed from pairs of electrons, so that the simplest superelectrons have $n^* \simeq n/2$, $m^* = 2m_e$ and $q = 2e$.

7.e [3 p] Based on the electrodynamics you have derived above, your understanding of how and why magnetic fields are created, and the general information given in this exercise, give a conceptual (physical) explanation of the phenomena illustrated in Figure 7.3.

The following may also be useful:
$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
$e = 1.602 \times 10^{-19}$ C
$m_e = 9.109 \times 10^{-31}$ kg
$\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}

8. Motor stunting

Rob Ouwersloot
PION winner 2010, 2011 and 2015
11 points

In this question we will look at the physics behind making front flips with a motorcycle. Assume the following things throughout the whole question:

- Any object will be two-dimensional (x and y) and infinitely thin in the z-direction. Any rotation will be around the z-axis, which points out of the paper. The x-axis runs to the right, the y-axis runs up.
- Ignore friction when calculating things, except for the static friction needed to make wheels roll or when stated otherwise.

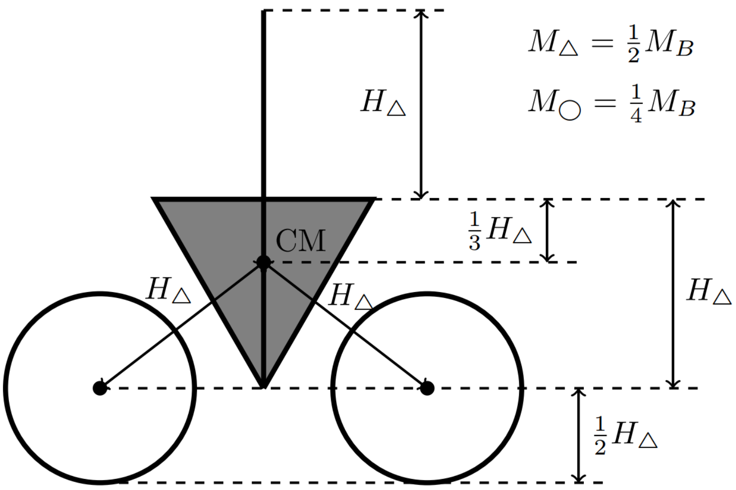

Figure 8.1 Approximation of a motorcycle with rider.

1. Calculating the moment of inertia

To start our analysis of making a front flip with a motorcycle, we need to know its moment of inertia. A very simplistic approximation of the shape of a motorcycle is given in Figure 8.1.

The motorcycle consists of three parts: two wheels and a central part. The central part is an equilateral triangle with height $H_\triangle$, a uniform mass density $\rho_\triangle$ and a total mass $M_\triangle$. The center of mass of the central part is indicated (CM). Both wheels have radius $R_\bigcirc = \frac{1}{2}H_\triangle$, mass $M_\bigcirc$ and a uniform mass distribution along the tire, which is infinitesimally thin. The body of the rider has length $L_B = 2H_\triangle$, mass M_B distributed evenly along the length and is also infinitesimally thin.

The moment of inertia of the wheels is quite trivial:

$$I_{\bigcirc} = \int\limits_{\text{wheel}} dm\ r^2 = R_{\bigcirc}^2 \int\limits_{\text{wheel}} dm = M_{\bigcirc} R_{\bigcirc}^2$$

The moment of inertia of the body is less trivial but not difficult:

$$I_B = \int\limits_{x=-L/2}^{x=+L/2} dx\ \frac{M}{L} x^2 = \frac{1}{12} M L^2.$$

However, there is a second, more elegant way to do this, which will become useful when calculating $I_{\triangle}$. It uses the following theorems, which follow from the definitions of the moment of inertia:

Theorem 1 (Parallel axis theorem) *Given the moment of inertia around an axis through the center of mass (CM) of an object, the moment of inertia of the same object with respect to a different but parallel axis is given by:*

$$I = I_{CM} + md^2$$

with m the mass of the object and d the (perpendicular) distance between the two axes.

Theorem 2 (Scaling of the moment of inertia) *If we scale the mass of an object by a factor c (not changing the dimensions), the new moment of inertia becomes $I' = cI$. If we scale the size of an object by a factor c (not changing the mass), the new moment of inertisa becomes $I' = c^2 I$.*

Let us now make a new object from the body, simply by putting two of them next to each other:

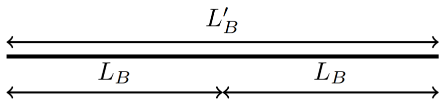

Figure 8.2

The new body has twice the mass and is twice as big, so the new moment of inertia is

$$I'_B = 2 \times 2^2 \times I_B = 8 I_B. \tag{8.1}$$

But the moment of inertia of the new body is also the sum of the two moments of inertia of the old body. Using the parallel axis theorem, the moment of inertia of one of the old bodies, can be expressed in terms of I_B, M_B and L_B. This gives a second equation for I'_B. Together with Equation 8.1 this can be solved for I_B in terms of M_B and L_B.

8.a [1 p] Show that this new method correctly recovers the moment of inertia of the body I_B.

For the central part of the bike we need to know the moment of inertia of a uniformly dense triangle. Calculating this by integration is quite cumbersome. Instead, it is suggested that you use the new method introduced in the previous section.

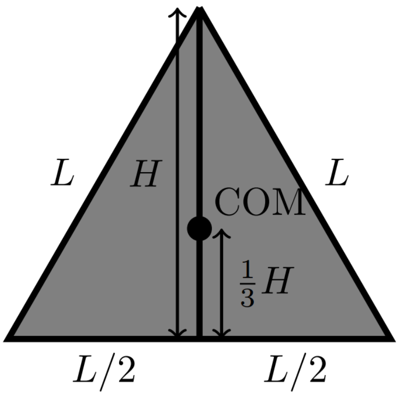

Figure 8.3

8.b [2 p] Show that the moment of inertia of an equilateral triangle with height $H_\triangle$ (and sides $L_\triangle = 2H_\triangle/\sqrt{3}$) and mass $M_\triangle$ is

$$I_\triangle = \frac{1}{9} M_\triangle H_\triangle^2.$$

You may use the fact that the center of mass of an equilateral triangle lies on the line connecting the top vertex (perpendicularly) with the middle of the base, at one third of the total length counted from the base.

8.c [1 p] Calculate the total moment of inertia I_M of the motorcycle with rider, with respect to their combined center of mass.

2. Jumping

Making a jump with a motorcycle is a quite straightforward process: take a ramp and drive to the top with a high enough speed.

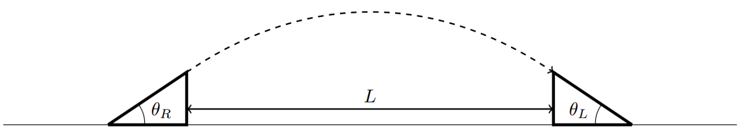

Figure 8.4

8.d [1 p] Given a jump gap L and a ramp angle θ_R, what is the minimum speed needed to clear the gap? Treat the motorcycle with rider as a point particle. Also determine the corresponding time of flight.

Landing on the other hand is not as easy and can be dangerous. To make it safer, the landing area is often also a ramp with a certain angle θ_L.

8.e [1 p] Give a physical reason why it is better for biker and motorcycle to try to land on a ramp instead of the flat ground.

From now on assume that $\theta_R = \theta_L = \theta$.

3. Flipping

Making an actual front or back flip on a motorcycle is a complicated process. We are going to make a very unphysical approximation and pretend that the rotation is provided only through braking the wheels. So we assume that, immediately after the wheels lose contact with the ramp, the rider brakes and the wheels immediately stop rotating. This causes a rotation of the motorcycle with rider which we can use to make our front flips.

If you did not manage to find the total moment of inertia I_M, you may use $I_M = 18 I_{\bigcirc}$.

8.f [2 p] For a fixed number of flips n, find the ramp angle θ that minimizes the initial speed v. Assume that the motorcycle has to land parallel to the landing ramp. You will find an equation for θ that is not solvable analytically. It is sufficient if you find the function $f(\theta, n)$ such that the solution to

$$\tan(\theta) = f(\theta, n)$$

is the optimal ramp angle.

4. Landing

When the motorcycle lands, the wheels will not be rotating. As long as the tangential velocity of the wheels is less than the velocity of the motor, the motor has no grip. This is potentially dangerous, as it means the rider cannot (effectively) brake the motor, so we want to know how long this dangerous phase is.

8.g [3 p] Assume we land the motor with speed v exactly parallel to the landing ramp. Assume we don't accelerate (or brake) the wheels and that the coefficient of kinetic friction (per wheel) is μ (which is much smaller than the coefficient of static friction). How long (in time) does the motorcycle slide before one of the wheels regains grip? Assume the landing ramp is long enough for this to happen on the ramp and express the time in v, θ, ϕ and μ.

9. The Bathtub Problem

Dr. Martin Rohde
TU Delft
7 points

Consider a cylindrical bathtub with a diameter of d meter, initially filled with water up to a height of h_0 meter. The bath can be emptied via a vertical pipe with a diameter of D meter and a length of H meter. The end of the vertical pipe is connected to the sewerage system. After bathing, the owner removes the plug but notices that the water level remains the same, i.e. no water is drained away. After inspection by a plumber he concludes that the end of the vertical pipe is clogged.

On $t = 0$, the plumber removes the blockage and water starts to drain from the bathtub. The vertical velocity of the water in the bathtub is v (being averaged over the cross-section of the tub) and the vertical velocity in the pipe is V (being averaged over the cross-section of the pipe). The top of the sewerage system is filled with air of ambient pressure, hence the water freely drops into the sewerage.

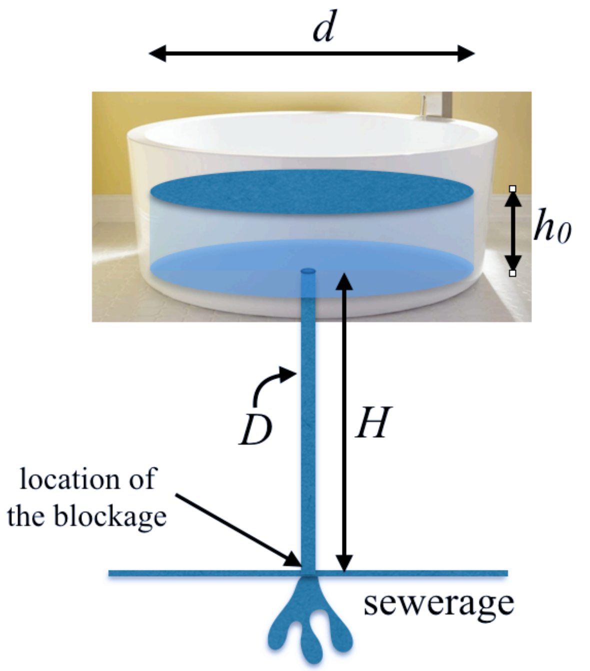

Figure 9.1

9.a [1 p] Express the change of the height of the water level in the bathtub, $h(t)$, as a function of the velocity of the water in the pipe, V.

9.b [1 p] Express the velocity of the water in the bathtub, v, in terms of the velocity in the vertical pipe, V. For this system, a few plausible assumptions can be made:

- The loss of mechanical energy due to the outflow of water is so small that the mechanical energy balance can be considered steady-state;
- Loss of mechanical energy by friction can be neglected;
- The density of the water, ρ, is constant throughout the system.

The above assumptions lead to the famous Bernoulli equation,

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz = \text{constant},$$

being a special form of the mechanical energy balance for fluids.
This equation holds for any point within the bathtub+pipe system.

9.c [5 p] Derive an expression for the time required to fully empty the bathtub as a function of the dimensions of the system with the help of the Bernoulli equation and your answers in a) and b).

10. A quantum one-dimensional wire

Prof. dr. Gary Steele
TU Delft
11 points

Consider an infinitely long wire made from a two-dimensional sheet of electrons in the $x-y$ plane, as illustrated below, with width w ($0 \leq x \leq w$). The sheet is filled with electrons with an average (number) density of n electrons/m^2. Electrons are confined within the sheet by an infinite hard-wall potential at its edges.

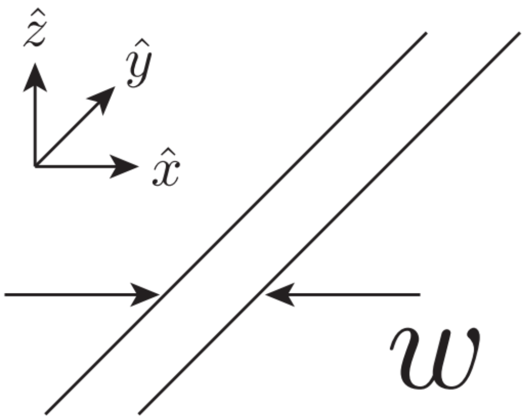

Figure 10.1

10.a [2 p] Solve the Schrödinger equation for the quantum wavefunction $\psi(x, y)$ for the electrons in your wire.

10.b [1 p] Sketch the dispersion relation in the y-direction for the solutions you found in (a).

10.c [2 p] In your research project, you will measure the properties of your wire in a dilution refrigerator that will cool your sample to a temperature of 50 mK. What constraint does this impose on the width w of your wire such that the motion of the electrons in it could be considered one-dimensional? Give a quantitative answer.

10.d [2 p] Consider a wire of width $w = 10$ nm. For such a value of w, what additional (quantitative) constraint is there on the electron density n such that the motion of the eletrons can be considered one-dimensional?

10.e [2 p] Now consider exposing your quantum wire to an electric field $\vec{E} = E_0\hat{z}$ oriented in the $\hat{z}$ (vertical) direction, perpendicular to the sheet of electrons. What is the effective magnetic field seen by electrons flowing in your wire?

10.f [2 p] Make a sketch of the dispersion relation of electrons in your wire including the effects of the vertical electric field. In your sketch, indicate the spin orientation of each band, and indicate with a formula the value of the splittings between the bands. Do electrons acquire a momentum from this electric field?

Formulae

Schrödinger equation:

$$\frac{\hat{\vec{p}}^2}{2m}\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y)$$

Momentum operators:

$$\hat{p}_x\psi(x,y) = -i\hbar\frac{\partial\psi}{\partial x}$$

$$\hat{p}_y\psi(x,y) = -i\hbar\frac{\partial\psi}{\partial y}$$

11. Ideal Qubits in canonical and micro-canonical ensemble

Dr. Misha Titov
Radboud Universiteit Nijmegen
9 points

Two-level quantum systems, especially those that can be isolated from the rest of the world, are called qubits. Such systems are the key elements of an important concept called quantum computation, which is based on quantum manipulations of qubits. A number of physical realizations of qubits have been proposed while an ideal one has yet to be found.

In this exercise you are suggested to describe an ensemble of ideal non-interacting qubits from a statistical physics point of view.

Think of the following two situations:

- non-interacting qubits coupled to a heat bath with the temperature T;
- fully isolated set of non-interacting qubits.

The first case corresponds to the so-called canonical ensemble of statistical mechanics while the second one corresponds to the microcanonical ensemble.

The microcanonical ensemble of statistical mechanics describes a fully isolated system that does not exchange heat or particles with anything around. Clearly, such an isolated system respects the conservation of energy E and the entropy S, while its temperature T, for example, is fluctuating. Realization of such a system would be ideal for quantum computation.

The canonical ensemble of statistical mechanics represents an equilibrium system that is brought into a contact with a heat reservoir (the heat 'bath'). In this case, the temperature is an external variable that is fixed by the bath while the internal energy of the system E is fluctuating. One often has to define the free energy $F(T) = \min_S(E(S) - TS)$ that is a conserved quantity in this ensemble. (The symbol $\min_S$ stands for the minimization with respect to the entropy).

As an example, consider an ensemble of N non-interacting qubits. Suppose that each qubit has two possible quantum states: one with the energy $\epsilon_0 = 0$ and another one with the energy $\epsilon_1 = \epsilon$.

Suppose first that the system is brought into a contact with a heat 'bath'. In this case the set of qubits must be described with the canonical ensemble.

11.a [1 p] Define the free energy of the system, $F = F(T, N)$ as the function of temperature T and the number of qubits N. (Use the relation $F = -k_B T \ln Z_N$ where Z_N is the partition function of the canonical ensemble and k_B is the Boltzmann constant). What happens with the free energy in the limit of large temperature $kT \gg \epsilon$? What is the value of the free energy at zero temperature? How would you explain or anticipate these results based on your intuition?

11.b [1.5 p] Find the (mean) entropy of the system, $S = -\frac{\partial F}{\partial T}$. What would you expect from the entropy in the limit $\epsilon \to 0$, i. e. for $\epsilon \ll kT$, and in the opposite limit of zero temperature $T \to 0$, i. e. for $\epsilon \gg kT$. How would you interpret the results obtained?

11.c [1.5 p] Find the average internal energy of the system $E = \langle E \rangle$, where the brackets stand for the averaging over the canonical statistical ensemble. What is the average energy per qubit in the limit of high temperature? How can you anticipate this result?

11.d [2.5 p] Find the relation between the variance of energy $\mathrm{var}\, E = \langle E^2 \rangle - \langle E \rangle^2$ and the heat capacity at a constant volume $C_V = T \frac{\partial S}{\partial T}$. Sketch the behaviour of C_V as a function of the dimensionless parameter ϵ/kT. Please explain the behaviour of C_V at large temperatures without making a calculation.

11.e [1 p] Consider now the situation of a microcanonical ensemble, when the system is fully isolated. What is the entropy of the system?

11.f [1.5 p] In the situation of the microcanonical ensemble, find the mean temperature of the system as a function of the conserved internal energy E. Sketch the averaged temperature of the system as a function of energy E for $E > N\epsilon/2$. Discuss the result obtained.

12. Feynman-Hellmann theorem and the hydrogen atom

Prof. dr. Peter van der Straten
Universiteit Utrecht
10 points

The Feynman-Hellmann theorem can be used to derive elements $\langle 1/r^s \rangle$ for the hydrogenic wavefunctions with s an integer, which can be used to evaluate the shift due to the fine structure interaction in the hydrogen atom.

Consider an Hamiltonian $\mathcal{H}(\mathrm{r}, \mathrm{q})$, eigenvalues $E_n(q)$ and eigenfunctions $u_n(r, q)$ that depend on a parameter q. The eigenvalues $E_n(q)$ are given by

$$E_n(q) = \int \mathrm{dr}\ \mathrm{u_n^*(r,q)}\mathcal{H}\mathrm{(r,q)u_n(r,q)}$$

12.a [2 p] Prove that the following relation for the derivative of $E(q)$ is correct:

$$\frac{\partial E_n(q)}{\partial q} = \left\langle \frac{\partial \mathcal{H}(\mathrm{r}, \mathrm{q})}{\partial q} \right\rangle$$

This relation is called the Feynman-Hellmann theorem.

This theorem can be applied to the hydrogenic wavefunctions by choosing the radial Hamiltonian of the hydrogen atom

$$\frac{\mathrm{d^2u_{El}(r)}}{\mathrm{dr^2}} + \frac{2m}{\hbar^2}\left[E - V_{\mathrm{eff}}(r)\right]u_{El}(r) = 0$$

with

$$V_{\mathrm{eff}}(r) \equiv -\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{l(l+1)\hbar^2}{2\mu r^2}$$

and its eigenvalues

$$E_n = -\frac{R_\infty Z^2}{n^2} = -\frac{mc^2\alpha^2 Z^2}{2n^2}\frac{\mu}{m}$$

with R_∞ the Rydberg constant, α the fine structure constant. Note that ℓ is related to n by the relation $n = n_r + \ell + 1$ with n_r the number of radial nodes.

12.b [1.5 p] Apply this theorem to find the expectation value $\langle 1/r \rangle$ by making an appropriate choice for q.

12.c [1.5 p] Find the expectation value $\langle 1/r^2 \rangle$ by making another choice for q.

For an expression for $\langle 1/r^3 \rangle$ proceed in a different way. We define the force on the electron as $F = \mathrm{d}V_{\text{int}}/\mathrm{d}r$.

12.d [2 p] Argue what the expectation value is for the force on the electron $\langle F(r) \rangle$ and provide your argument(s).

12.e [1 p] Find the relation for $\langle 1/r^3 \rangle$.

12.f [2 p] Show why this does not work for $\ell = 0$.