Contents

1. A gentle persuasion

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11 points

1.a [3 p] We start by deriving the Lagrangian of the two point particles, assuming a vanishing external gravitational energy at $y = 0$ and small oscillations of the pendulums. In polar coordinates the kinetic energy is given by the usual expression:

$$T_{kin} = \frac{1}{2} m l^2 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$$

with $\dot{\varphi}_{1,2} \equiv \frac{\mathrm{d}\varphi_{1,2}}{\mathrm{d}t}$ representing the angular velocities of the two pendulums. The potential energy due to all gravitational interactions amounts to

$$\begin{aligned} V_{grav} &= -mg(y_1 + y_2) - \frac{G_N m^2}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \\ &= -mgl(\cos\varphi_1 + \cos\varphi_2) - \frac{G_N m^2}{l\sqrt{(1 + \sin\varphi_2 - \sin\varphi_1)^2 + (\cos\varphi_2 - \cos\varphi_1)^2}} \end{aligned}$$

with G_N denoting Newton's constant. For small oscillations we can expand this up to second order in the angular displacements:

$$\begin{aligned} V_{grav} &\approx -mgl\Big[2 - \frac{1}{2}(\varphi_1^2 + \varphi_2^2)\Big] - \frac{G_N m^2}{l(1 + \varphi_2 - \varphi_1)} \\ &\approx -mgl\Big[2 - \frac{1}{2}(\varphi_1^2 + \varphi_2^2)\Big] - \frac{G_N m^2}{l}\Big[1 - (\varphi_2 - \varphi_1) + (\varphi_2 - \varphi_1)^2\Big] \end{aligned}$$

The resulting approximated Lagrangian $L = T_{kin} - V_{grav}$ of the system then reads

$$L(\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2) \approx \frac{1}{2} m l \Big[l(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + (\mathrm{g}' - \mathrm{g})(\varphi_1^2 + \varphi_2^2) - 2\mathrm{g}'\varphi_1\varphi_2 + \mathrm{g}'\varphi_1 - \mathrm{g}'\varphi_2 + C \Big]$$

where

$$\mathrm{g}' = \frac{2mG_N}{l^2}$$
$$C = 4\mathrm{g} + \mathrm{g}'$$

1.b [1 p] The equations of motion for $\varphi_{1,2}$ are obtained from the Lagrangian of part (a) through the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) - \frac{\partial L}{\partial \varphi_1} = 0 \quad \Rightarrow \quad \ddot{\varphi}_1 + \frac{\mathrm{g}}{l}\varphi_1 - \frac{\mathrm{g}'}{l}(\varphi_1 - \varphi_2) - \frac{\mathrm{g}'}{2l} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) - \frac{\partial L}{\partial \varphi_2} = 0 \quad \Rightarrow \quad \ddot{\varphi}_2 + \frac{\mathrm{g}}{l}\varphi_2 - \frac{\mathrm{g}'}{l}(\varphi_2 - \varphi_1) + \frac{\mathrm{g}'}{2l} = 0$$

1.c [1 p] Before trying to solve these coupled differential equations, we first determine the equilibrium angle ε in terms of g and g′. The equilibrium configuration corresponds to the conditions $\ddot{\varphi}_{1,2} = \dot{\varphi}_{1,2} = 0$. From the equations of motion it follows that for equilibrium

$$\begin{aligned} \mathrm{g}\varphi_1^{\mathrm{eq}} - \mathrm{g}'(\varphi_1^{\mathrm{eq}} - \varphi_2^{\mathrm{eq}}) &= \mathrm{g}'/2 \\ \mathrm{g}\varphi_2^{\mathrm{eq}} + \mathrm{g}'(\varphi_1^{\mathrm{eq}} - \varphi_2^{\mathrm{eq}}) &= -\mathrm{g}'/2 \\ \Rightarrow \varepsilon \equiv \varphi_1^{\mathrm{eq}} = -\varphi_2^{\mathrm{eq}} &= \frac{\mathrm{g}'}{2\mathrm{g} - 4\mathrm{g}'} \end{aligned}$$

1.d [3 p] The eigenmodes of the system are obtained by considering the angular variables $\varphi_\pm \equiv \varphi_1 - \varepsilon \pm (\varphi_2 + \varepsilon)$, for which the equations of motion decouple according to

$$\begin{aligned} \ddot{\varphi}_+ &= -\frac{\mathrm{g}}{l}\varphi_+ \equiv -\omega_+^2\varphi_+ \\ \ddot{\varphi}_- &= -\frac{\mathrm{g} - 2\mathrm{g}'}{l}\varphi_- \equiv -\omega_-^2\varphi_- \end{aligned}$$

1.e [3 p] Now we can solve the equations of motion given the initial conditions at time $t = 0$

$$\begin{aligned} \varphi_1(0) &= \varepsilon + \varphi_0 \\ \varphi_2(0) &= -\varepsilon \\ \dot{\varphi}_1(0) = \dot{\varphi}_2(0) &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} \varphi_\pm(0) &= \varphi_0 \\ \dot{\varphi}_\pm(0) &= 0 \end{aligned}$$

The solutions are trivially given by

$$\varphi_\pm(t) = \varphi_0 \cos(\omega_\pm t).$$

Introducing

$$\begin{aligned} \bar{\omega} &\equiv \frac{\omega_+ + \omega_-}{2} \\ \Delta_\omega &\equiv \frac{\omega_+ - \omega_-}{2} \end{aligned}$$

we obtain the following solutions for each of the pendulums:

$$\varphi_1(t) = \varepsilon + \frac{\varphi_0}{2}\big[\cos(\omega_+ t) + \cos(\omega_- t)\big] = \varepsilon + \varphi_0 \cos(\bar{\omega} t)\cos(\Delta_\omega t)$$

$$\varphi_2(t) = -\varepsilon + \frac{\varphi_0}{2}\big[\cos(\omega_+ t) - \cos(\omega_- t)\big] = -\varepsilon - \varphi_0 \sin(\bar{\omega} t)\sin(\Delta_\omega t)$$

If $\mathrm{g}' \ll \mathrm{g}$, then $\bar{\omega} \approx \sqrt{\mathrm{g}/l}$ and $\Delta_\omega \approx \varepsilon\sqrt{\mathrm{g}/l}$ with $\varepsilon \approx \mathrm{g}'/(2\mathrm{g}) \ll 1$. Therefore, initially only the first pendulum is swinging and bit by bit energy is being transferred gravitationally to the second pendulum, until at time $t = \pi/(2\Delta_\omega)$ the situation is completely reversed and only the second pendulum is swinging.

Let's plug in the following numerical input: $m = 1$ kg, $l = 1$ m, $\mathrm{g} = 9.81$ m s^{-2} and $G_N = 6.67 \times 10^{-11}\mathrm{m}^3$ kg^{-1} s^{-2}. In that case the switch will take place in approximately $\pi/(2\Delta_\omega) \approx \pi\sqrt{\mathrm{g}l}/\mathrm{g}' = \pi\sqrt{\mathrm{g}l^5}/(2mG_N) = 7.38 \times 10^{10}$ s $= 2334$ years!

2. Optical Cavities: The Fabry-Perot Etalon

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8 points

2.a [0.5 p] Using Snell's law twice, we find $\theta_e = \theta_i$:

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_t}{n_i} = \frac{n}{1} = n$$
$$\frac{\sin\theta_t}{\sin\theta_e} = \frac{n_e}{n_t} = \frac{1}{n}$$

Substituting $\sin\theta_t$:

$$\sin\theta_t = \sin\theta_e \frac{1}{n}$$
$$n = \frac{\sin\theta_i}{\sin\theta_t} = \frac{\sin\theta_i}{\sin\theta_e \frac{1}{n}}$$
$$\sin\theta_i = \sin\theta_e \Rightarrow \theta_e = \theta_i$$

2.b [1 p] The phase difference $\delta = k\Delta$ where Δ is the optical path length difference (OPD). $\Delta = 2nl\cos\theta_t$. The OPD between two reflected beams (as seen in Figure 2.1) is given by:

$OPD = \mathrm{ab} + \mathrm{bc} - \mathrm{ad}$
$\mathrm{ab} = \mathrm{bc}$
$\mathrm{ab} + \mathrm{bc} = \frac{2\mathrm{nl}}{\cos\theta_\mathrm{t}}$
$\mathrm{ad} = \mathrm{ac}\sin\theta_\mathrm{i}$
$\frac{\mathrm{ac}}{2} = \frac{\sin\theta_t}{\mathrm{ab}} = \frac{nl\sin\theta_t}{\cos\theta_t}$
$\sin\theta_i = n\sin\theta_t$
$\mathrm{ad} = \frac{2\mathrm{nl}\sin^2\theta_\mathrm{t}}{\cos\theta_\mathrm{t}}$
Therefore the OPD $= \frac{2nl - 2nl\sin^2\theta_t}{\cos\theta_t}$
$= \frac{2nl(1-\sin^2\theta_t)}{\cos\theta_t} = \frac{2nl\cos^2\theta_t}{\cos\theta_t}$
$\mathrm{OPD} = 2nl\cos\theta_t$
For constructive interference OPD $= 2nl\cos\theta_t = 2m\pi$ where m is an integer number.

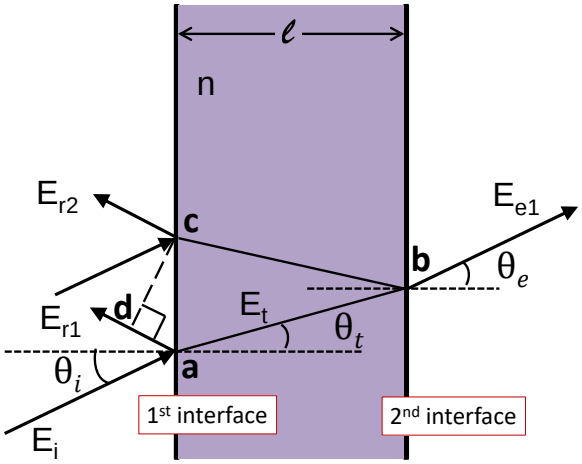


Figure 2.1 Optical path length difference for reflected light

2.c [1 p]

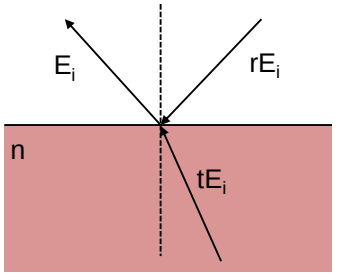
- The incident ray build up of the reflected and transmitted rays is shown in Figure 2.2a
- The Stokes Relation figure considering all rays is shown in Figure 2.2b

In both figures, r' and t' denote the reflection and transmission coefficients when the ray is incident from the medium (so originating from tE_i).

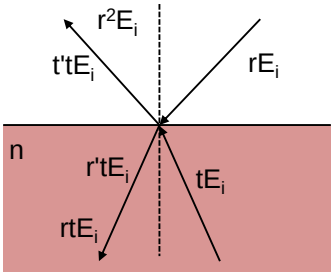
2.d [1 p] To build up the incident ray in Figure 2.1:
$E_i = (r^2 + t't)E_i$
and for the reflected ray in the medium (non-existant in the original figure) in Figure 2.2a:
$0 = (r't + tr)E_i$
thus we get
$tt' = 1 - r^2$
$r = -r'$.
There is a π phase change for every reflection in the second media.

2.e [1.5 p] The total transmitted amplitude can be written as

$$E_{tr} = \sum_{m=1}^{\infty} E_{em}$$
$$= tt'E_{in} + rr'e^{i\delta}tt'E_{in} + (rr')^2e^{2i\delta}tt'E_{in} + ... = tt'E_{in}(1 + rr'e^{i\delta} + (rr')^2e^{2i\delta} + ...)$$
$$= tt'E_{in}\sum_{j=1}^{\infty}(rr'e^{i\delta})^j$$



(a) Reflection and transmission at an interface between two media with different refractive indices reversed to build up the incident ray.



(b) Reflection and transmission at an interface between two media with different refractive indices used for deriving the Stokes relations.

Figure 2.2

using the summation of geometric series

$$E_{tr} = \frac{tt' E_{in}}{1 - rr' e^{i\delta}}$$

2.f [1 p] Taking the modulus of the amplitude expression obtained above we get

$$\begin{aligned} I_{tr} = |E_{tr}|^2 &= \frac{tt' E_{in}}{1 - rr' e^{i\delta}} \cdot \frac{tt' E_{in}^*}{1 - rr' e^{-i\delta}} \\ &= |E_{in}|^2 \frac{t^2 t'^2}{(1 - rr' e^{i\delta})(1 - rr' e^{-i\delta})} \\ &= I_{in} \frac{TT'}{1 + RR' - 2\sqrt{RR'}\cos\delta} \end{aligned}$$

where we have used $T = t^2$, $T' = t'^2$, $R = r^2$, $R' = r'^2$, $e^{ix} + e^{-ix} = 2\cos x$

2.g [0.5 p] For energy conservation $R + T = 1$.

2.h [1 p] The fringes are a group of concentric rings called the Airy's rings.

2.i [0.5 p] One would see two concentric patterns corresponding to each wavelength and having the amplitude of each of the wavelengths.

3. Long range molecular interactions

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PION winner 2016
11 points

3.a [1 p] The Pauli exchange interaction term is given by:

$$K = \langle \psi_A | \hat{H} | \psi_B \rangle$$

Both molecular wave functions decay exponentially so also the exchange integral will decay exponentially. The electrostatic potential decays with a power of R, meaning that this becomes much larger than the exchange interaction.

3.b [3 p] The electrostatic potential energy between the two molecules is given by:

$$V = \sum_i \sum_j \frac{q_i q_j}{\mathbf{R} + (\mathbf{r_j} - \mathbf{r_i})}$$

$$V = \sum_i \sum_j \frac{q_i q_j}{\sqrt{(R + (r_{j,z} - r_{i,z}))^2 + (r_{j,x} - r_{i,x})^2 + (r_{j,y} - r_{i,y})^2}}$$

$$V = \sum_i \sum_j \frac{q_i q_j}{\sqrt{R^2 + 2R(r_{j,z} - r_{i,z}) + (r_{j,z} - r_{i,z})^2 + (r_{j,x} - r_{i,x})^2 + (r_{j,y} - r_{i,y})^2}}$$

Now use the Taylor approximation:

$$\frac{1}{\sqrt{(1+x)}} \approx 1 - \frac{x}{2} + \frac{3x^2}{2}$$

Only take into account terms with powers of R of -3 or higher:

$$V = \sum_i \sum_j \frac{q_i q_j (1 - \frac{(r_{j,z} - r_{i,z})}{R} - \frac{|r|^2}{2R^2} + \frac{3(r_{j,z} - r_{i,z})^2}{2R^2})}{R}$$

$$V = \sum_i \sum_j \frac{q_i q_j}{R} - \sum_i \sum_j \frac{q_i q_j (r_{j,z} - r_{i,z})}{R^2} +$$
$$\frac{1}{2} \sum_i \sum_j \frac{q_i q_j (2(r_{j,z} - r_{i,z})^2 - (r_{j,x} - r_{i,x})^2 - (r_{j,y} - r_{i,y})^2)}{R^3}$$

$$V = \frac{\sum_i q_i \sum_j q_j}{R} + \frac{\sum_i q_i r_{i,z} \sum_j q_j}{R^2} - \frac{\sum_i q_i \sum_j q_j r_{j,z}}{R^2} +$$
$$\frac{1}{2} \sum_i \sum_j \frac{q_i q_j (3(r_{j,z}^2 + r_{i,z}^2) - |\mathbf{r_j}|^2 - |\mathbf{r_i}|^2 - \mathbf{2 r_{j,z} r_{i,z}} + \mathbf{r_{j,x} r_{i,x}} + \mathbf{r_{j,y} r_{i,y}})}{R^3}$$

Using the definitions of the dipole moment $\sum_i q_i r_i$ and the quadrupole moment, this expression becomes:

$$V(R) = \frac{q_A q_B}{R} + \frac{q_B \mu_{A,z} - q_A \mu_{B,z}}{R^2} + \frac{2 q_A Q_{B,zz} + 2 q_B Q_{A,zz} + \mu_A \mathbf{T} \boldsymbol{\mu}_\mathbf{B}}{R^3}$$

3.c [2 p] The first order energy correction is given by:

$$E_0^{(1)} = \langle \psi_{A,0}^{(0)} \psi_{B,0}^{(0)} | \frac{q_B}{R^2} \mu_{A,z} | \psi_{A,0}^{(0)} \psi_{B,0}^{(0)} \rangle$$
$$E_0^{(1)} = \frac{q_B}{R^2} \langle \psi_{B,0}^{(0)} | \psi_{B,0}^{(0)} \rangle \langle \psi_{A,0}^{(0)} | \mu_{A,z} | \psi_{A,0}^{(0)} \rangle$$
$$E_0^{(1)} = \frac{q_B}{R^2} \langle \psi_{A,0}^{(0)} | \mu_{A,z} | \psi_{A,0}^{(0)} \rangle$$

The matrix element of the dipole moment operator is the quantum mechanical expectation value of the dipole moment. This term is therefore exactly the classical term.

$$E_0^{(2)} = \frac{q_B^2}{R^4} \sum_{m>0} \frac{|\langle \psi_{A,0}^{(0)} | \mu_{A,z} | \psi_{A,m}^{(0)} \rangle|^2}{E_{A,0}^{(0)} - E_{A,m}^{(0)}}$$

So this gives a value of $\alpha_{A,zz}$ of:

$$\alpha_{A,zz} = 2 \sum_{m>0} \frac{|\langle \psi_{A,0}^{(0)} | \mu_{A,z} | \psi_{A,m}^{(0)} \rangle|^2}{E_{A,m}^{(0)} - E_{A,0}^{(0)}}$$

This $\alpha_{A,zz}$ is the zz-component of the polarizability tensor. The second order energy correction gives the interaction between the charge and the induced dipole moment.

3.d [3 p] The first order energy correction is given by:

$$E_0^{(1)} = \langle \psi_{A,0}^{(0)} \psi_{B,0}^{(0)} | \frac{\hat{\mu}_A \mathbf{T} \hat{\boldsymbol{\mu}}_\mathbf{B}}{R^3} | \psi_{A,0}^{(0)} \psi_{B,0}^{(0)} \rangle$$

T does not act on the wave functions. Again this terms with wave function A and B can be separated. This gives:

$$E_0^{(1)} = \frac{1}{R^3} \langle \psi_{A,0}^{(0)} | \hat{\mu}_A | \psi_{A,0}^{(0)} \rangle \mathbf{T} \langle \boldsymbol{\psi}_{\mathbf{B},\mathbf{0}}^{(\mathbf{0})} | \hat{\boldsymbol{\mu}}_\mathbf{B} | \psi_{B,0}^{(0)} \rangle$$

This is again exactly the classical expression, where the expectation values of the dipole moments replace the classical dipole moments. The second order energy corrections are now given by:

$$E_0^{(2)} = \sum_{n,m,n+m>0} \frac{|\langle \psi_{A,0}^{(0)} \psi_{B,0}^{(0)} | \frac{\hat{\mu}_A \mathbf{T} \hat{\boldsymbol{\mu}}_\mathbf{B}}{R^3} | \psi_{A,m}^{(0)} \psi_{B,n}^{(0)} \rangle|^2}{E_{A,0}^{(0)} + E_{B,0}^{(0)} - E_{A,m}^{(0)} - E_{B,n}^{(0)}}$$

$$E_0^{(2)} = \frac{1}{R^6} \sum_{n,m,n+m>0} \frac{|\langle \psi_{A,0}^{(0)} | \hat{\mu}_A | \psi_{A,m}^{(0)} \rangle \mathbf{T} \langle \boldsymbol{\psi}_{\mathbf{B},\mathbf{0}}^{(\mathbf{0})} | \hat{\boldsymbol{\mu}}_\mathbf{B} | \psi_{B,n}^{(0)} \rangle|^2}{E_{A,0}^{(0)} + E_{B,0}^{(0)} - E_{A,m}^{(0)} - E_{B,n}^{(0)}}$$

This can be separated into three terms by splitting up the sum. Two terms are acquired by setting m or n to 0. The third terms is the remaining term where both m and n are nonzero.

If m or n is set to 0:

$$E_0^{(2)} = \frac{1}{R^6} \sum_m \frac{|\langle \psi_{A,0}^{(0)} | \hat{\mu}_A | \psi_{A,m}^{(0)} \rangle \mathbf{T} \langle \boldsymbol{\psi}_{\mathbf{B},\mathbf{0}}^{(\mathbf{0})} | \hat{\boldsymbol{\mu}}_\mathbf{B} | \psi_{B,0}^{(0)} \rangle|^2}{E_{A,0}^{(0)} - E_{A,m}^{(0)}}$$

This reduces to a term with the polarizability of molecule A and the dipole moment of molecule B. This is therefore the interaction of the dipole moment and the induced dipole moment.

If m and n are both nonzero than the terms of molecules A and B remain mixed. This is a purely quantum mechanical interaction which has no classical equivalent. This interaction term is called the dispersion interaction or VanderWaals interaction!

3.e [2 p]

- The major interaction here is the charge-induced dipole interaction, which goes with R^{-4}.
- The major interaction here is the dispersion interaction, which goes with R^{-6}.
- Benzene has no charge or a dipole moment but it does have a quadrupole moment. The quadrupole-quadrupole interaction is the most important, going with R^{-5}. This could be found out by extrapolating formula (1) further to higher powers of R^{-6}
- The major interaction here is a dipole-dipole interaction, which goes with R^{-3}

4. Cesium hyperfine structure and atomic clocks

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12 points

4.a [2 p] Cs is a Alkali metal. The Alkali atoms (first column of the period table) are effectively single electron systems. The valence electron can be excited from the s orbital in the ground state to an excited p orbital (B). This level has a different energy from the ground state due to the imperfect screening of the multiply charged nucleus in Cs by the other electrons, called the quantum defect. This excited state itself is split into two levels due to the spin-orbit interaction (C) (also called fine structure). By closer inspection we see that the ground state is further split due to the coupling with the nuclear spin (A). Also the electronically excited states show hyperfine structure.

To determine the relative magnitudes of these splittings, we can start with splitting A, which is given in the question to be about 9 GHz, so order 10^{10} Hz. The spin-orbit splitting is a factor of about μ_B/μ_N larger, which is about 2000 (1836). This gives for splitting C a magnitude of 10^{13} Hz. The electronic transition for all Alkali atoms is in the optical (remember the yellow Sodium street lamps), corresponding to a frequency of 10^{14} to 10^{15} Hz.

Energy splitting	Physical mechanism giving rise to splitting	Energy splitting size, order of magnitude n (in 10^n Hz)
A	Electronically excited state. Difference in energy between the valence electron in the s and p orbital, due to the quantum defect.	$n = 14 - 15$
B	Finestructure: coupling of electron magnetic moment due to spin withmagnetic moment due to orbital angular momentum.	$n = 13$
C	Hyperfinestructure: coupling of total angular momentum withthe magnetic moment of nuclear spin.	$n = 10$

4.b [3 p]

– The ground state, because it has a single electron in an s orbital, has orbital angular momentum $L = 0$. Because $S = \frac{1}{2}$. $J = L+S = \frac{1}{2}$. Since the nuclear spin $I = 7/2$, this leads to two hyperfine levels with quantum numbers $F = 3$ and $F = 4$. With aid of the given formula, we first find g_J, and from that the two g_F values: $g_J = 2 \Rightarrow g_{F=3} = -\frac{1}{4}$; $g_{F=4} = \frac{1}{4}$. For the two hyperfine states the Zeeman shifts are thus given by $\mu_B m_F g_F B_z = \frac{1}{4} m_F \mu_B B_z$ for the F=4 level, and $-\frac{1}{4} m_F \mu_B B_z$ for the F=3 level. The difference in energy between neighbouring hyperfine m_F levels is thus $\frac{1}{4}\mu_B B_z$.

– In a weak magnetic field the energy levels m_F splitted from hyperfine level F are given by: $-F \leq m_F \leq F$. So there will be $2F+1$ splitted levels for each hyperfine level. In a strong magnetic field J and I are decoupled and the hyperfine structure is just a perturbation on the splitting due to the electron spin. The states at low magnetic field labelled by m_F, now correspond to states with m_J and m_I, where $m_J = \pm\frac{1}{2}$ and $-7/2 \leq m_I \leq 7/2$, so there are $2I + 1 = 8$ sublevels levels for the two possible values of m_J.

	Weak magnetic field	Strong magnetic field
Higher group of energy levels	# levels $= 2F + 1 = 9$	# levels $= 2I + 1 = 8$
Lower group of energy levels	# levels $= 2F + 1 = 7$	# levels $= 2I + 1 = 8$

4.c [3 p] The magnetic force on the atoms is in strong fields proportional to $-\frac{\partial B}{\partial z} m_J$. The atoms with $m_J = \frac{1}{2}$ are forced in the direction of decreasing magnetic field B and atoms with $m_J = -\frac{1}{2}$ are forced in the direction of increasing magnetic field. These are so called low field seeking and high field seeking states respectively.

The energy levels of the atoms correspond to different orientations of the electron spins with respect to the local magnetic field. If the magnetic field changes, the spins of the atoms tend to remain in the same relative orientation (ie. the same quantum state). This means that the atoms will usually rotate along with the magnetic field when it changes direction. This is what is meant by the adiabatic condition: the magnetic field change is sufficiently slow so that the atoms can follow.

Atoms in low-field seeking states will remain in low-field seeking states and thus deflect towards lower field strengths (and vice versa), regardless of the direction of the magnetic field.

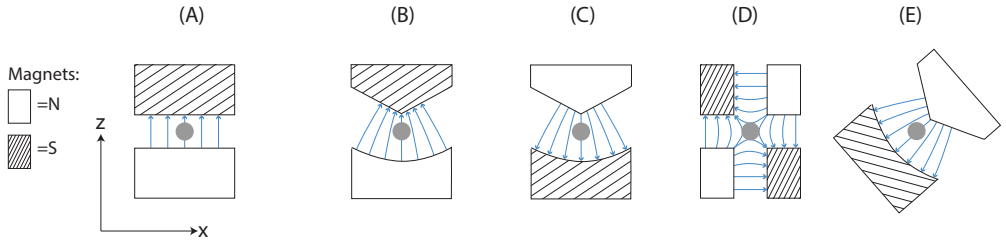

Figure 4.1

A sketch of the field lines can be used to find the high and low field regions in the indicated magnet configurations: the magnetic field is strong where the lines lie close together and weak where they are far apart. (A) causes no deflection (only the potential energy and thus the velocity of the atoms is changed). (B) and (C) cause a vertical deflection. (D) has a focusing or defocusing effect. (E) causes a deflection with both a horizontal and a vertical component.

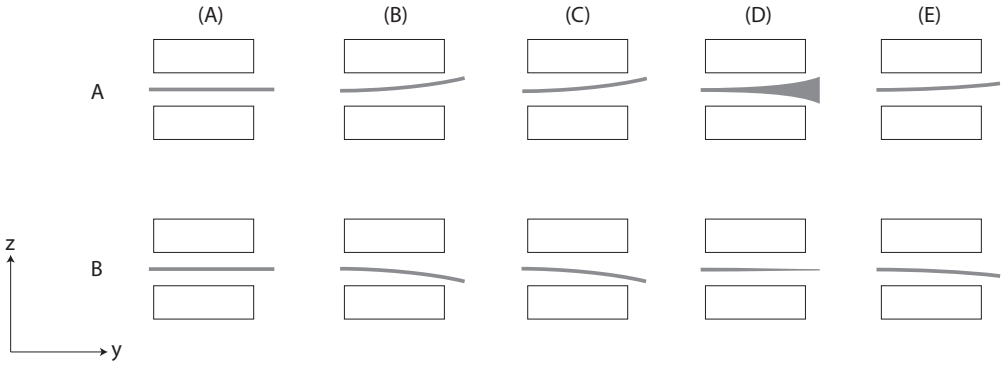

Figure 4.2

4.d [4 p]

(1).
The oven produces beams of Cs-133 atoms in the electronic ground states, so it is not hyperfine-state selective, atoms in all hyperfine states $(3, m_F)$ and $(4, m_F)$ are produced.

(2).
In the strong magnetic field regime, I and J are decoupled and $m_F = m_J + m_I$. A strong magnet seperates the beam based on $J = L + S = \frac{1}{2}$. Two beams corresponding to $m_j = \pm\frac{1}{2}$ emerge after the magnets. From the geometry of the magnets we see that the magnetic field strength decreases in the positive z-direction, this means that only the atoms in the high field seeking states ($m_J = -\frac{1}{2}$) reach location 2. Since no level crossings are allowed for the atoms with $F = 3$ these are all the m_F states, while for $F = 4$ only the $m_F = -4$ qualifies. Thus the resulting atomic states are $(3, m_F)$ and $(4, -4)$

(3).
The microwave field drives the transition $(3, 0) \rightarrow (4, 0)$, so after this field the resulting states are: $(3, -3), (3, -2), (3, -1), (4, 0), (3, 1), (3, 2), (3, 3), (4, -4)$

(4).
Considering the geometry of the magnets again, only the atoms in low field seeking states arrive at this location. So only atoms in state $(4, 0)$.

	Location (1)	(2)	(3)	(4)
(F, m_F)	(3,m_F) (4,m_F)	(3,m_F) (4,-4)	(3,-3),(3,-2),(3,-1),(4,0), (3,1),(3,2),(3,3),(4,-4)	(4,0)

5. Some beer physics

Dr. Arie van Houselt
University of Twente
10 points

5.a [3 p] When the surface of separation undergoes an infinitesimal displacement, the length of a segment of the normal lying between the displaced and undisplaced surface is δl.

A volume element between the two surfaces is then given by $\delta l \mathrm{dA}$, with dA a surface element. With p_1 and p_2 the pressures in the two media, the work needed for the change in volume is

$$\int (-p_1 + p_2)\delta l \mathrm{dA}$$

The work needed to change the surface area is proportional to the change δA in surface area and is given by $\gamma \delta A$, with γ the surface tension. In equilibrium the total work

$$\delta W = -\int (p_1 - p_2)\delta l \mathrm{dA} + \gamma \delta \mathrm{A} = 0$$

With r' and r'' are the principal radii of curvature at a given point of the surface, the arc differentials dA' and dA'' on the surface in its principal curvature sections are increased to

$$\mathrm{dA}' \rightarrow \frac{\mathrm{dA}'(\mathrm{r}' + \delta \mathrm{l})}{\mathrm{r}'} = \mathrm{dA}'(1 + \frac{\delta \mathrm{l}}{\mathrm{r}'})$$

and

$$\mathrm{dA}'' \rightarrow \frac{\mathrm{dA}''(\mathrm{r}'' + \delta \mathrm{l})}{\mathrm{r}''} = \mathrm{dA}''(1 + \frac{\delta \mathrm{l}}{\mathrm{r}''})$$

Hence a surface element $\mathrm{dA} = \mathrm{dA}'\mathrm{dA}''$ becomes after displacement:

$$\mathrm{dA}_{\mathrm{new}} = \mathrm{dA}'(1 + \frac{\delta \mathrm{l}}{\mathrm{r}'})\mathrm{dA}''(1 + \frac{\delta \mathrm{l}}{\mathrm{r}''} \approx \mathrm{dA}(1 + \frac{\delta \mathrm{l}}{\mathrm{r}'} + \frac{\delta \mathrm{l}}{\mathrm{r}''})$$

thus it changes by

$$\Delta A = \delta l \mathrm{dA}(\frac{1}{\mathrm{r}'} + \frac{1}{\mathrm{r}''})$$

The total change in area for the surface of separation is thus:

$$\delta A = \int \delta l(\frac{1}{r'} + \frac{1}{r''})\mathrm{dA}$$

Hence we obtain for equilibrium:

$$\int \delta l\Big[(p_1 - p_2) - \gamma(\frac{1}{r'} + \frac{1}{r''})\Big]\mathrm{dA}$$

Since this must hold for every δl, the expression between brackets must equal zero:

$$\delta P = \gamma(\frac{1}{r'} + \frac{1}{r''})$$

5.b [2 p] See the sketch in Figure 5.1.

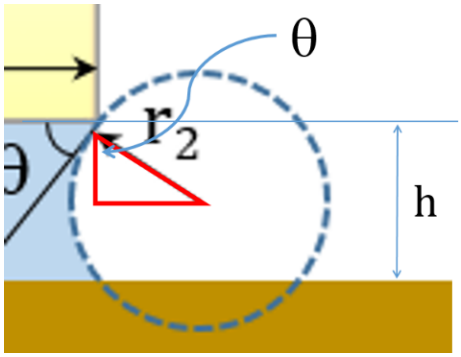

Figure 5.1

The indicated angle equals θ. So

$$\cos\theta = \frac{\frac{h}{2}}{r_2}$$

$$r_2 = \frac{h}{2\cos\theta}$$

Use the Laplace equation

$$\Delta P = \gamma\Big(\frac{1}{r'} + \frac{1}{r''}\Big)$$

with r'= r_1 = 3cm; r'' = $-r_2 = -\frac{h}{2\cos\theta}$. Thus

$$\Delta P = \gamma\Big(\frac{1}{r_1} - \frac{2\cos\theta}{h}\Big)$$

The cappilary force F_c equals $\Delta P \times \pi r_1^2$, which should balance $F_g = -mg$. Thus

$$\Delta P = \gamma\Big(\frac{1}{r_1} - \frac{2\cos\theta}{h}\Big) = -\frac{mg}{\pi r_1^2}$$

$$\frac{1}{r_1} - \frac{2\cos\theta}{h} = -\frac{mg}{\gamma\pi r_1^2}$$

$$-\frac{2\cos\theta}{h} = -\frac{mg}{\gamma\pi r_1^2} - \frac{1}{r_1} = -\frac{mg + \gamma\pi 1}{\gamma\pi r_1^2}$$

$$h = \frac{2\gamma\pi r_1^2\cos\theta}{mg + \gamma\pi r_1} = 9.13\times 10^{-4} m \approx 1mm$$

5.c [1 p] The presence of these proteins will lower the surface tension. Keeping the Laplace pressure the same, h must decrease to balance the reduced surface tension ⇒ maximum film thickness reduces. Alternatively: the surface tension changes the numerater in the formula derived in a more drastically than the denominator, hence h will decrease.

5.d [3 p] B is formed in a one-to-one ratio from A, and similarly C from B, thus:

$$\frac{\mathrm{dB}}{\mathrm{dt}} = k_1 A - k_2$$

To find A:

$$\int_{A_0}^{A_t} \frac{1}{A}\mathrm{dA} = -\mathrm{k}_1 \int_0^{\mathrm{t}} \mathrm{dt}$$

$$\ln A - \ln A_0 = \ln\frac{A}{A_0} = -k_1 t$$

$$A(t) = A_0 e^{-k_1 t}$$

To find B:

$$\int_0^B 1\mathrm{dB}' = \mathrm{k}_1 \int_0^{\mathrm{t}} \mathrm{A(t')dt'} - \mathrm{k}_2 \int_0^{\mathrm{t}} \mathrm{dt}'$$

$$= k_1 A_0 \int_0^t e^{-k_1 t'}\mathrm{dt}' - \mathrm{k}_2 \int_0^{\mathrm{t}} \mathrm{dt}$$

$$B = k_1 A_0 [\frac{e^{-k_1 t'}}{-k_1}]_0^t - k_2 t = -A_0(e^{-k_1 t} - 1) - k_2 t$$

$$B = A_0(1 - e^{-k_1 t}) - k_2 t$$

Or find B using $A_0 = A + B + C$ and $A = A_0 e^{-k_1 t}$, $C = k_2 t$

At time t_m:

$$\frac{\mathrm{dB}}{\mathrm{dt}} = 0$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big(A_0(1 - e^{-k_1 t}) - k_2 t\Big)\Big|_{t=t_m}$$

$$= A_0 k_1 e^{-k_1 t_m} - k_2 = 0$$

$$e^{-k_1 t_m} = \frac{k_2}{A_0 k_1}$$

$$\Rightarrow -k_1 t_m = \ln \frac{k_2}{A_0 k_1}$$

$$t_m = \frac{1}{k_1} \ln \frac{A_0 k_1}{k_2}$$

At this time, the amount of A is given by:

$$A(t = t_m) = A_0 e^{-k_1 t_m} = A_0 e^{-k_1 \frac{1}{k_1} \ln \frac{A_0 k_1}{k_2}}$$

$$= A_0 e^{\ln \frac{k_2}{A_0 k_1}} = A_0 \frac{k_2}{A_0 k_1}$$

$$A(t = t_m) = \frac{k_2}{k_1}$$

5.e [1 p] For a zero order reaction (constant reaction rate) the reactants disappear linearly with time. Only the first 25 min (≈7 % of the total reaction time) are nonlinear, (with an amount of B lower than in a first order extrapolation) thus the assumption of a first order process was justified.

6. Dark Matter and Gravitational Lensing

Dr. Søren Larsen
Radboud University
10 points

6.a [2 p]

– Mass distribution in the galaxy: For circular velocity v, the mass M within a radius R is

$$M(R) = \frac{v^2 R}{G}$$

This follows from the acceleration in circular motion

$$a = \frac{v^2}{R}$$

which must be equal to the acceleration on a test particle due to the gravity of the central mass

$$F = ma = G\frac{Mm}{R^2}$$

so

$$a = G\frac{M}{R^2}$$

That is

$$\frac{v^2}{R} = G\frac{M}{R^2}$$

so

$$M = \frac{v^2 R}{G}$$

So if v is independent of R, then we have that $M \propto R$.

– The mass of the halo is:

$$M(R) = \frac{v^2 R}{G}$$

Thus the derivative of the density is

$$\frac{\mathrm{dM}}{\mathrm{dR}} = \frac{v^2}{G}$$

so the amount matter in a thin shell of thickness dR is

$$\mathrm{dM} = \frac{\mathrm{v}^2}{\mathrm{G}}\,\mathrm{dR}$$

For density $\rho(R)$, we also have

$$\mathrm{dM} = 4\pi \mathrm{R}^2 \rho\,\mathrm{dR}$$

and therefore

$$\frac{v^2}{G} = 4\pi R^2 \rho$$

so that

$$\rho(R) = \frac{v^2}{4\pi G R^2}$$

6.b [3 p] The classical calculation gives the acceleration of the photon as

$$a = \frac{GM_L}{r^2}$$

The perpendicular component of a is

$$a_\perp = a\frac{r_{\mathrm{min}}}{r} = \frac{GM_L r_{\mathrm{min}}}{r^3}$$

We have

$$\begin{aligned} r^2 &= r_{\mathrm{min}}^2 + d^2 \\ &= r_{\mathrm{min}}^2 + (ct)^2 \end{aligned}$$

so

$$a_\perp = \frac{GM_L r_{\mathrm{min}}}{(r_{\mathrm{min}}^2 + (\mathrm{ct})^2)^{3/2}}$$

The perpendicular velocity acquired is then

$$v_\perp = \int_{-\infty}^{\infty} a_\perp \mathrm{dt}$$

$$= \int_{-\infty}^{\infty} \frac{GM_L r_{\mathrm{min}}}{(r_{\mathrm{min}}^2 + (\mathrm{ct})^2)^{3/2}} \mathrm{dt}$$

We use the hint:

$$\int \frac{1}{(a+bx^2)^{3/2}}\mathrm{dx} = \frac{\mathrm{x}}{\mathrm{a}\sqrt{\mathrm{a}+\mathrm{bx}^2}}$$

with $a = r_{\mathrm{min}}^2$ and $b = c^2$ and find

$$v_\perp = GM_L r_{\mathrm{min}} \left.\frac{\mathrm{t}}{\mathrm{r}_{\mathrm{min}}^2\sqrt{\mathrm{r}_{\mathrm{min}}^2+\mathrm{c}^2\mathrm{t}^2}}\right|_{\mathrm{t}=-\infty}^{\mathrm{t}=\infty} = 2\frac{GM_L}{cr_{\mathrm{min}}}$$

so the angle is

$$\alpha = \frac{v_\perp}{c} = \frac{2GM_L}{r_{\mathrm{min}}\mathrm{c}^2}$$

6.c [2 p] If $D_{SL} \gg D_{LO}$ then the light rays from the star can be considered parallel until they get deflected, so the radius of the Einstein ring is equal to the deflection angle, $\alpha = \theta_E$. Then

$$\theta_E = \frac{4GM_L}{r_{\mathrm{min}}\mathrm{c}^2}$$

Assume that deflection occurs instantaneously near L; then we have

$$r_{\mathrm{min}} = \theta_{\mathrm{E}}\mathrm{D}_{\mathrm{LO}}$$

Then

$$\theta_E = \frac{4GM_L}{\theta_E D_{LO}c^2}$$

so

$$\theta_E^2 = \frac{4GM_L}{D_{LO}c^2}$$

6.d [2 p] If the mass of a single MACHO is M_L and the mass of the halo is M_H, then the number of MACHOS is

$$N = M_H/M_L$$

Each MACHO has an Einstein ring with area

$$A_E = \pi\theta_E^2 = \frac{4\pi GM_L}{D_{LO}c^2}$$

so the total area of all Einstein rings is

$$A_{\rm tot} = {\rm N}A_{\rm E} = \frac{{\rm M_H}}{{\rm M_L}}\frac{4\pi {\rm G M_L}}{{\rm D_{LO}c^2}} = \frac{4\pi G M_H}{D_{LO}c^2}$$

The sky has a total area of 4π sr, so the optical depth is

$$\tau_L = \frac{GM_H}{D_{LO}c^2}$$

which is indeed independent of M_L. Filling in $M_H = 4 \times 10^{11} M_\odot \approx 8 \times 10^{41}$ kg and $d = 10^4$ pc $\approx 3 \times 10^{20}$ m, we get $\tau_L = 1.9 \times 10^{-6}$.

6.e [1 p] The angular velocity is $v_\theta = v/D_{LO}$. Hence, the time to move across the Einstein ring is

$$t = 2\theta_E/v_\theta = 2\frac{D_{LO}}{v}2\sqrt{\frac{GM_L}{D_{LO}c^2}} = 4\sqrt{\frac{D_{LO}GM_L}{v^2c^2}}$$

For $M_L = 1M_\odot$, $d = 10^4$ pc $= 3.08 \times 10^{20}$ m, and $v = 2 \times 10^5$ m/s, we get $t = 1.35 \times 10^7$ s $= 156$ days.

Constants:

1 pc $= 3.08 \times 10^{16}$ m
Speed of light: $c = 3 \times 10^8$ m s^{-1}
Gravitational constant: G $= 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
1 M$_\odot$ $= 1.989 \times 10^{30}$ kg

7. Superelectrodynamics

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Radboud University
10 points

7.a [2 p] The emf in the RL circuit of Figure 2 is

$$\varepsilon = IR + L\frac{dI}{dt}$$

In a constant magnetic field, no current flows in the circuit and so $\varepsilon = 0$. However, when the field is switched off, the magnetic flux threading the circuit changes and an emf is induced according to the flux rule. If $R = 0$, as for the superconducting ring, then

$$\varepsilon = -\frac{d\phi}{dt} = -\frac{d(\oint \mathbf{B}\cdot d\mathbf{a})}{dt} = -A\frac{dB}{dt} = L\frac{dI}{dt}$$

where $\mathbf{B}$ is the magnetic field and A is the area of the circuit loop (ring) perpendicular to the field direction.

$$\Rightarrow \qquad L\frac{dI}{dT} + A\frac{dB}{dt} = 0$$

and integrating with respect to time gives

$$LI + AB = \text{constant}. \tag{7.1}$$

$LI + AB$ is the total magnetic flux threading the circuit (ring) and Equation 7.1 shows that if B decreases, I will increase to exactly compensate the change in flux, and vice versa.

7.b [1.5 p] In a perfect conductor, the equation of motion for electrons gives

$$\mathbf{E} = -\frac{m_e}{e}\frac{\partial \mathbf{v}}{\partial t}. \tag{7.2}$$

($\tau \to \infty$ means that the final term in equation (1) is zero.) The time derivative of the current density is

$$\frac{\partial \mathbf{J}}{\partial t} = -ne\frac{\partial \mathbf{v}}{\partial t} \tag{7.3}$$

so substituting Equation 7.3 in Equation 7.2 gives

$$\mathbf{E} \;=\; \frac{m_e}{ne^2}\,\frac{\partial \mathbf{J}}{\partial t}. \tag{7.4}$$

Applying Maxwell's equations:

From Faraday, $\boldsymbol{\nabla}\times\mathbf{E} \;=\; -\dfrac{\partial \mathbf{B}}{\partial t}$,

so taking the curl of Equation 7.4 gives

$$\frac{m_e}{ne^2}\left(\boldsymbol{\nabla}\times\frac{\partial \mathbf{J}}{\partial t}\right) \;=\; -\frac{\partial \mathbf{B}}{\partial t}. \tag{7.5}$$

The Maxwell-Ampère law is $\boldsymbol{\nabla}\times\mathbf{B} \;=\; \mu_0\mathbf{J} \;+\; \mu_0\epsilon_0\,\dfrac{\partial \mathbf{E}}{\partial t}$

and we are considering weak, quasistatic fields (even 'switching off' gives a relatively small dB/dt) so we can assume the second term is negligible. Taking the time derivative:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\frac{\partial \mathbf{B}}{\partial t} \;=\; -\nabla^2\,\frac{\partial \mathbf{B}}{\partial t} \;=\; \mu_0\boldsymbol{\nabla}\times\mathbf{J}.$$

(the divergence term is zero because $\boldsymbol{\nabla}\cdot\mathbf{B}=0$). Substituting Equation 7.5

$$\nabla^2\,\frac{\partial \mathbf{B}}{\partial t} \;=\; \frac{m_e}{\mu_0 ne^2}\,\frac{\partial \mathbf{B}}{\partial t} \;=\; \frac{1}{\lambda^2}\,\frac{\partial \mathbf{B}}{\partial t}. \tag{7.6}$$

7.c [1.5 p] The London equation is

$$\boldsymbol{\nabla}\times\frac{m^*}{n^*q^2}\,\mathbf{J}_s = -\mathbf{B}. \tag{7.7}$$

Taking the curl of both sides of the Maxwell-Ampère law gives $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B} \;=\; \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \nabla^2\mathbf{B} \;=\; \mu_0\boldsymbol{\nabla}\times\mathbf{J}$.

again neglecting the term in $\partial\mathbf{E}/\partial t$. If $\mathbf{J} = \mathbf{J}_s$, i.e. if we are considering a superconductor, we can simply substitute Equation 7.7 and we find

$$\nabla^2\mathbf{B} \;=\; \frac{1}{\lambda_L^2}\,\mathbf{B}. \tag{7.8}$$

7.d [2 p] Equation (6) has the solution

$$\frac{\partial \mathbf{B}_y}{\partial t}(x) \;=\; \frac{\partial \mathbf{B}_y}{\partial t}(0)\;\exp\left(-\frac{x}{\lambda}\right) \tag{7.9}$$

describes the exponential decay of $\partial\mathbf{B}/\partial t$ with distance inside the surface of a perfect conductor. λ is the characteristic length for the decay. From the definition of λ, it is easy to show that it has the units of length, and a quick calculation using the n given in the exercise yields $\lambda = 16.8$ nm as a 'typical' value. Deeper than a few times λ into the conductor we have $\partial\mathbf{B}/\partial t \to 0$, so $\mathbf{B}$ must be constant.

That is, Equation 7.6 and Equation 7.9 (Equation 3 in the exercise) tell us that magnetic field (magnetic flux) can vary within a small distance of the perfect conductor's surface, characterised by the length λ, but that $\mathbf{B}$ must always remain constant in the interior of a perfect conductor.

Equation (7) has the solution,

$$\mathbf{B}_y(x) \;=\; \mathbf{B}_y(0)\exp\left(-\frac{x}{\lambda_L}\right) \tag{7.10}$$

which tells us that the magnetic field $\mathbf{B}$ decays exponentially with distance inside the surface, with characteristic decay length λ_L, and that $\mathbf{B}$ must not be simply constant, but must be always equal to zero in the interior of a superconductor.

7.e [3 p] For the first sphere:

When it is cooled it becomes a perfect conductor and must now obey equation (3) (Equation 7.6 above), which states that deeper than a few times λ into the conductor, the magnetic field must always remain constant with the same strength and direction. Removing the external magnetic field creates a time-variation of $\mathbf{B}$ and, in accordance with Faraday's law of electromagnetic induction, current is generated in the conducting sphere. By Lenz's law (also expressed by Equation 7.1 above), this current acts to oppose the change that causes it, and it creates magnetic field inside the sphere to cancel the effect of switching off the external field. The current flows close to the surface of the sphere, within a depth of a few times λ, so that a constant magnetic field is maintained deeper inside the sphere. Because the sphere is a perfect conductor, the induced current persists without loss, and the magnetic field in the interior of the sphere is maintained indefinitely.

For the second sphere:

This sphere becomes superconducting when it is cooled and must obey Equation (5) (Equation 7.8 above), which states that the magnetic field in the interior of

the superconductor must be zero. Field can penetrate up to a few times λ_L into the sphere, and equation (4) says that, in a superconductor, magnetic field leads to circulating supercurrent in a direction so as to oppose $\mathbf{B}$. This supercurrent therefore flows in the surface region of the sphere, and acts to perfectly cancel the magnetic field in the interior. When the external magnetic field is removed, further currents are induced (in accordance with Faraday's law) to oppose the change, and the balance of surface supercurrents is always such that $B = 0$ is maintained in the interior.

8. Motor stunting

Rob Ouwersloot
PION winner 2010, 2011 and 2015
11 points

8.a [1 p] Due to symmetry it is obvious that the COM of the new body lies in between the two old bodies. So the distance of the new COM to both old COM's is $L_B/2$. This gives

$$I'_B = \left(I_B + M_B\left(\frac{L_B}{2}\right)^2\right) + \left(I_B + M_B\left(\frac{L_B}{2}\right)^2\right)$$
$$= 2I_B + \frac{1}{2}M_B L_B^2.$$

Combining with the formula for a body with twice the mass and that is twice as big, so the new moment of inertia is

$$I'_B = 2 \times 2^2 \times I_B = 8I_B.$$
$$8I_B = 2I_B + \tfrac{1}{2}M_B L_B^2$$
$$6I_B = \tfrac{1}{2}M_B L_B^2$$
$$I_B = \tfrac{1}{12}M_B L_B^2$$

which is indeed correct.

8.b [2 p] Approach 1: Using a scaling argument (either approach is correct and can yield the full points)

We make a new equilateral triangle out of four identical equilateral triangles, by putting them together as indicated in the picture:

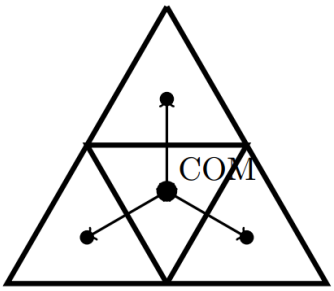

Figure 8.1

The COM of the middle and the grand triangle coincide. The distance between the COM of the grand triangle and the three outer sub triangles is two times $\frac{1}{3}H$, as the distance between COM and any side of the equilateral triangle is $\frac{1}{3}H$. The moment of inertia of the grand triangle is:

$$I'_{\triangle} = 4 \times 2^2 \times I_{\triangle} = 16I_{\triangle}$$

as the mass has scaled by a factor of 4 and the size by a factor of 2. The moment of inertia of the grand triangle is also (by composition)

$$I'_{\triangle} = I_{\triangle} + 3(I_{\triangle} + M_{\triangle}(\tfrac{2}{3}H)^2) = 4I_{\triangle} + \tfrac{4}{3}M_{\triangle}H_{\triangle}^2.$$

Equating these gives

$$\begin{aligned} 16I_{\triangle} &= 4I_{\triangle} + \tfrac{4}{3}M_{\triangle}H_{\triangle}^2 \\ 12I_{\triangle} &= \tfrac{4}{3}M_{\triangle}H_{\triangle}^2 \\ I_{\triangle} &= \tfrac{1}{9}M_{\triangle}H_{\triangle}^2 \end{aligned}$$

Approach 2: Using integration (either approach is correct and can yield the full points)

The moment of inertia of a triangle can also be determined by integration.

$$I_{\triangle} = \int_{\triangle} dr^2 \rho r^2$$

Let us take the COM as the origin and the x-axis parallel to the base. The difficulty then lies in deciding which integration interval is dependent on the other. If we take $y \in [\frac{-1}{3}H, \frac{2}{3}H]$, we get $x \in [(y/H - \frac{2}{3})L/2, (-y/H + \frac{2}{3})L/2]$. This becomes:

$$I_{\triangle} = \rho \int_{y=\frac{-1}{3}H}^{\frac{1}{3}H} dy \int_{x=(y/H-\frac{2}{3})L/2}^{(-y/H+\frac{2}{3})L/2} dx x^2 + y^2$$

$$= \rho \int_{y=\frac{-1}{3}H}^{\frac{2}{3}H} dy (\frac{2}{3} - \frac{y}{H})Ly^2 + \frac{2}{3}(\frac{2}{3} - \frac{y}{H})^3(\frac{L}{2})^3$$

Introducing $q = y/H$, with $dy = Hdq$, gives

$$I_{\triangle} = LH\rho \int_{q=\frac{-1}{3}}^{\frac{2}{3}} dq(\frac{2}{3}-q)H^2q^2 + \frac{1}{12}(\frac{2}{3}-q)^3L^2$$

$$= LH\rho\Big\{H^2\Big[\frac{2}{9}q^3 - \frac{1}{4}q^4\Big]_{-\frac{1}{3}}^{\frac{2}{3}} + \frac{1}{12}L^2\Big[-\frac{1}{4}(\frac{2}{3}-q)^4\Big]_{-\frac{1}{3}}^{\frac{2}{3}}\Big\}$$

$$= LH\rho\Big\{H^2\Big[\frac{1}{12}(\frac{2}{3})^4 + \frac{11}{12}(\frac{1}{3})^4\Big] + \frac{1}{12}L^2\Big[0 - \frac{-1}{4}\Big]\Big\}$$

Using $L_{\triangle} = 2H_{\triangle}/\sqrt{3}$ and $M_{\triangle} = \frac{1}{2}LH\rho$

$$I_{\triangle} = 2M_{\triangle}\Big\{H^2\Big[\frac{27}{12\cdot 81}\Big] + \frac{1}{48}\frac{4}{3}H^2\Big\}$$

$$= 2M_{\triangle}H^2\Big\{\frac{1}{36} + \frac{1}{36}\Big\} = \frac{4}{36}M_{\triangle}H^2 = \frac{1}{9}M_{\triangle}H_{\triangle}^2$$

as expected.

8.c [1 p] Due to symmetry is obvious that the x-coordinate of the COM_M coincides with the x-coordinate of the $\mathrm{COM}_{\triangle}$. The y-coordinate is given by:

$$y_{COM} = \frac{1}{M_M}\left(M_B y_B + M_{\triangle}y_{\triangle} + M_{\bigcirc_1}y_{\bigcirc_1} + M_{\bigcirc_2}y_{\bigcirc_2}\right)$$

$$= \frac{1}{2M_B}\left(M_B(y_{\triangle} + \tfrac{1}{3}H_{\triangle}) + \tfrac{1}{2}M_B y_{\triangle} + 2\cdot\tfrac{1}{4}M_B(y_{\triangle} - \tfrac{2}{3}H_{\triangle})\right.$$

$$= \frac{1}{2}\left(y_{\triangle} + \tfrac{1}{3}H_{\triangle} + \tfrac{1}{2}y_{\triangle} + \tfrac{1}{2}y_{\triangle} - \tfrac{1}{3}H_{\triangle}\right)$$

$$= y_{\triangle}.$$

(Or, in words, because the body is twice as close (in y-direction) than the wheels (together), but also twice as heavy, their combined COM coincides with the COM of the central part. And then it is trivial that the total COM also coincides with the COM of the central part.)

Now the total moment of inertia becomes (it is obvious that both wheels contribute equally):

$$
\begin{aligned}
I_M &= I_\triangle + I_B + 2I_\bigcirc \\
&= \tfrac{1}{9} \times \tfrac{1}{2} M_B \times H_\triangle^2 + \tfrac{1}{12} \times M_B \times (2H_\triangle)^2 + M_B \times \left(\tfrac{1}{3} H_\triangle\right)^2 \\
&\quad + 2 \times \tfrac{1}{4} M_B \times (\tfrac{1}{2} H_\triangle)^2 + 2\tfrac{1}{4} M_B \times H_\triangle^2 \\
&= \left(\tfrac{1}{18} + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{8} + \tfrac{1}{2}\right) M_B H_\triangle^2 \\
&= \left(\tfrac{4}{72} + \tfrac{24}{72} + \tfrac{8}{72} + \tfrac{9}{72} + \tfrac{36}{72}\right) M_B H_\triangle^2 \\
&= \tfrac{81}{72} M_B H_\triangle^2 = \tfrac{9}{8} M_B H_\triangle^2
\end{aligned}
$$

8.d [1 p] In the vertical direction, the intial speed is $v_{y,i} = v\sin(\theta_R)$ and the acceleration is g. Thus, setting the initial (and thus also the final) height at 0, we get

$$
\begin{aligned}
y(t) &= vt\sin(\theta_R) - \tfrac{1}{2} g t^2 \\
\Rightarrow y(t_{\mathrm{TOF}}) &= v t_{\mathrm{TOF}} \sin(\theta_R) - \tfrac{1}{2} g t_{\mathrm{TOF}}^2 = 0 \\
t_{\mathrm{TOF}} &= \frac{2v\sin(\theta_R)}{g}
\end{aligned}
$$

In the horizontal direction, we only have the initial speed $v_x = v\cos(\theta_R)$, so

$$
\begin{aligned}
x(t_{\mathrm{TOF}}) = L &= v\cos(\theta_R) t_{\mathrm{TOF}} = \frac{2v^2}{g} \sin(\theta_R)\cos(\theta_R) \\
\Rightarrow v &= \sqrt{\frac{Lg}{2\sin(\theta_R)\cos(\theta_R)}} = \sqrt{\frac{Lg}{\sin(2\theta_R)}}.
\end{aligned}
$$

Which yields for t_{TOF}

$$
t_{\mathrm{TOF}} = \frac{2\sin(\theta_R)}{g} \sqrt{\frac{Lg}{2\sin(\theta_R)\cos(\theta_R)}} = \sqrt{\frac{2L\sin(\theta_R)}{g\cos(\theta_R)}} \left(= \sqrt{\frac{2L\tan(\theta_R)}{g}} \right)
$$

8.e [1 p] Landing on a ramp means the speed is more parallel to the landing plane and thus that the perpendicular speed is smaller. The perpendicular speed has to become zero (the motorcycle doesn't bounce away) and the impact of this velocity change is much smaller on ramp. In the ideal case the ramp would be parallel to the landing velocity.

8.f [2 p] Initially the wheels rotate at angular velocity

$$
\omega_W = \frac{v}{R_\bigcirc} = \frac{2v}{H_\triangle}
$$

Conservation of angular momentum dictates

$$2\omega_{\bigcirc} I_{\bigcirc} = \omega_M I_M$$

so the angular velocity of the motorcycle with biker becomes:

$$\omega_M = \frac{2I_{\bigcirc}}{I_M}\omega_{\bigcirc} = \frac{2\frac{1}{16}M_B H_{\triangle}^2}{\frac{9}{8}M_B H_{\triangle}^2}\frac{2v}{H_{\triangle}} = \frac{2v}{9H_{\triangle}}.$$

The total angle which we have to rotate through Θ is:

$$2\theta + 2n\pi = \Theta = \omega_M t_{\mathrm{TOF}} = \frac{2v}{9H_{\triangle}}\frac{2v\sin(\theta)}{g} = \frac{4v^2\sin(\theta)}{9H_{\triangle}g}$$

This gives for v^2

$$v^2 = \frac{9H_{\triangle}g}{2}\left(\frac{\theta + n\pi}{\sin(\theta)}\right).$$

We want to minimize v, but as v is positive, we might as well minimize v^2, which has a simpler formula. So differentiating with respect to θ and equating to 0:

$$\begin{aligned}
\frac{\partial}{\partial\theta}v^2 = \frac{\partial}{\partial\theta}\left(\frac{\theta + n\pi}{\sin(\theta)}\right) &= 0 \\
\frac{\sin(\theta) - \cos(\theta)(\theta + n\pi)}{\sin^2(\theta)} &= 0 \\
\Rightarrow \sin(\theta) &= \cos(\theta)(\theta + n\pi) \\
\tan(\theta) &= \theta + n\pi
\end{aligned}$$

I.e. $f(\theta, n) = \theta + n\pi$. This has no analytical solution. A sketch or plot will reveal that the solution will be close to (but smaller than) $\pi/2$ and will become closer with increasing n.

8.g [3 p] The total normal force on the motorcycle is equal to the component of the gravitational force perpendicular to the ramp. Thus $F_N = \cos(\theta)F_Z$. The acceleration of the motorcycle due to the gravitational force is equal to the parallel component: $a_Z = g\sin(\theta)$. The total frictional force is $F_W = \mu F_N = \mu\cos(\theta)mg$, so the nett acceleration is

$$a = g\sin(\theta) - \frac{F_W}{m} = g\sin(\theta) - \mu\cos(\theta)g = g\left(\sin(\theta) - \mu\cos(\theta)\right)$$

so the parallel speed is

$$v(t) = v + gt\left(\sin(\theta) - \mu\cos(\theta)\right).$$

Simultaneously, the wheels will start to rotate. The nett torque on each wheel depends on the friction, which depends on the normal force. The total normal force is known, but the normal force per wheel is less trivial. We observe that the motorcycle does not rotate (as a whole) so there is no nett torque around its center of mass. The gravitational force acts directly on the COM, so this cannot provide any torque. This leaves the friction on both wheels and the normal force on both wheels to balance each other (torque-wise).

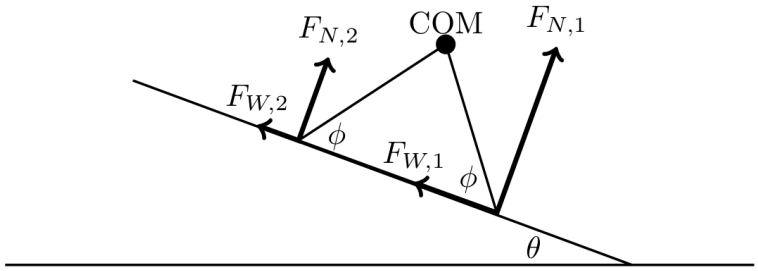

Figure 8.2

The distance from the center of mass to the contact points is irrelevant, but let's denote it with L for now. This gives us 2 equations. First of all, the nett torque

$$\sum \vec{\tau} = \sum \vec{F} \times \vec{r} = 0 \quad \Rightarrow \quad \sum \tau = \sum \pm Fr \sin(\text{interior angle}) = 0$$

(with sign dependent on wether the torque points clockwise or counter-clockwise) is zero:

$$\begin{aligned} \sin\left(\tfrac{\pi}{2} - \phi\right) F_{N,2} + \sin\left(\pi - \phi\right) F_{W,2} + \sin\left(\phi\right) F_{W,1} &= \sin\left(\tfrac{\pi}{2} - \phi\right) F_{N,1} \\ \cos\left(\phi\right) F_{N,2} + \sin\left(\phi\right) \mu F_{N,2} + \sin\left(\phi\right) \mu F_{N,1} &= \cos\left(\phi\right) F_{N,1} \\ \left(\cos\left(\phi\right) + \mu \sin\left(\phi\right)\right) F_{N,2} &= \left(\cos\left(\phi\right) - \mu \sin\left(\phi\right)\right) F_{N,1} \\ \Rightarrow \quad F_{N,2} &= \frac{1 - \mu \tan(\phi)}{1 + \mu \tan(\phi)} F_{N,1} \end{aligned}$$

The other equation we have, is that the sum of the normal forces on the wheels is equal to the total normal force F_N.

$$\begin{aligned} F_N = F_{N,1} + F_{N,2} &= \left(1 + \frac{1 - \mu \tan(\phi)}{1 + \mu \tan(\phi)}\right) F_{N,1} \\ \cos(\theta) mg = \left(\frac{1 + \mu \tan(\phi) + 1 - \mu \tan(\phi)}{1 + \mu \tan(\phi)}\right) F_{N,1} &= \frac{2}{1 + \mu \tan(\phi)} F_{N,1} \\ \Rightarrow \quad F_{N,1} &= \tfrac{1}{2} \left(1 + \mu \tan(\phi)\right) \cos(\theta) mg \end{aligned}$$

This results in the following torque on the first wheel

$$\tau_1 = F_{W,1}\tfrac{1}{2}H_{\triangle} = \tfrac{1}{4}\mu\left(1+\mu\tan(\phi)\right)\cos(\theta)M_B g H_{\triangle}$$

which results in an angular acceleration

$$\alpha = \frac{\tau}{I_W} = \frac{\frac{1}{4}\mu\left(1+\mu\tan(\phi)\right)\cos(\theta)M_B g H_{\triangle}}{\frac{1}{16}M_B H_{\triangle}^2} = \frac{4\left(1+\mu\tan(\phi)\right)\mu g\cos(\theta)}{H_{\triangle}}.$$

The tangential velocity of the wheel(s) is equal to $v_W(t) = \omega_W R = \alpha t\frac{1}{2}H_{\triangle} = 4\mu\left(1+\mu\tan(\phi)\right)gt\cos(\theta)$. This is equal to the parallel speed at:

$$\begin{aligned} v + gt\left(\sin(\theta) - \mu\cos(\theta)\right) &= 4\left(1+\mu\tan(\phi)\right)\mu g t\cos(\theta) \\ v &= gt\left(5\mu\cos(\theta) + 4\mu^2\tan(\phi)\cos(\theta) - \sin(\theta)\right) \\ \Rightarrow t &= \frac{1}{5\mu\cos(\theta) + 4\mu^2\tan(\phi)\cos(\theta) - \sin(\theta)}\frac{v}{g} \end{aligned}$$

9. The Bathtub Problem

Dr. Martin Rohde
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7 points

9.a [1 p] This relation can be derived from the mass balance for the system. The total mass in the system is equal to

$$M = \frac{1}{4}\pi\rho(D^2H + d^2h(t))$$

The change of mass with respect to time equals the outflow of water, hence

$$\frac{dM}{dt} = -\frac{1}{4}\pi D^2\rho V,$$

where V denotes the average velocity of the water in the vertical pipe. The above leads to

$$\frac{d}{dt}\Big\{\frac{1}{4}\pi\rho(D^2H + d^2h(t))\Big\}$$
$$= \frac{d}{dt}\Big\{\frac{1}{4}\pi\rho d^2h(t)\Big\} = -\frac{1}{4}\pi D^2\rho V$$

which can be simplified to

$$\frac{dh(t)}{dt} = -(\frac{D}{d})^2 V \tag{9.1}$$

9.b [1 p] We know that the mass of water leaving the bathtub per unit of time should be equal to the mass of water entering the vertical pipe per unit of time (if not, mass would accumulate or disappear at the interface between bathtub and pipe). Hence

$$\phi_m^{leaving} = \phi_m^{entering}\ (\frac{kg}{s}).$$

In terms of velocities, we get

$$\rho v \cdot \frac{1}{4}\pi d^2 = \rho V \cdot \frac{1}{4}\pi D^2,$$

so that we get the desired relation

$$v = \left(\frac{D}{d}\right)^2 V \tag{9.2}$$

Note that this relation can also be derived from the answer of question a), since $v = -\dfrac{dh(t)}{dt}$!

9.c [5 p] For this question, we take two points in the water volume, being:

- a point at the surface of the water in the bathtub (with $v = v$, $p = p_0$, $z = H + h(t)$)
- a point at the exit of the vertical pipe (with $v = V$, $p = p_0$, $z = 0$)

Since we know that $\frac{p}{\rho} + \frac{1}{2}v^2 + gz =$ constant throughout the entire volume, we know that

$$\frac{p_0}{\rho} + \frac{1}{2}v^2 + g\left(H + h(t)\right) = \frac{p_0}{\rho} + \frac{1}{2}V^2 + g \cdot 0. \tag{9.3}$$

Note that the pressures at both points are the same since the water freely falls into the sewerage system. Combining Equation 9.3 and Equation 9.2 we get

(9.4)

This result qualitatively is correct: when the height of water in the tub decreases, the velocity of the water decreases as we know from experience. Substituting Eq. Equation 9.4 into Equation 9.1 gives a differential equation describing the relation between the height $h(t)$ and time t:

$$\frac{dh(t)}{dt} = -\sqrt{\frac{2g\left(H + h(t)\right)}{(d/D)^4 - 1}} = -\sqrt{C_1\left(H + h(t)\right)}.$$

Solving this differential equation gives

$$\frac{dh(t)}{\sqrt{H + h(t)}} = -\sqrt{C_1}dt \rightarrow 2\sqrt{H + h(t)} = -\sqrt{C_1}t + C_2.$$

From the intial condition, $t = 0 : h(0) = h_0$, we find that $C_2 = 2\sqrt{H + h_0}$. Hence,

$$2\sqrt{H + h(t)} = -\sqrt{C_1}t + 2\sqrt{H + h_0}.$$

The time t_e at which the bathtub is emptied is determined by the fact that $h(t_e) = 0$. So we get

$$t_e = \frac{2\sqrt{H + h_0} - 2\sqrt{H}}{\sqrt{C_1}} = \left(\sqrt{H + h_0} - \sqrt{H}\right)\sqrt{2\frac{(d/D)^4 - 1}{g}}$$

10. A quantum one-dimensional wire

Prof. dr. Gary Steele
TU Delft
11 points

10.a [2 p] Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = E\Psi$$

Choose coordinates such that one edge of the wire is at x = 0 and the other is at x = w. In this case, V(x,y) is given by:

$$V(x,y) = \begin{cases} 0 & \text{for } 0 < \text{x} < \text{w} \\ \infty & \text{otherwise} \end{cases}$$

Note that V(x,y) is separable (in fact, it depends only on x). This allows us to write:

$$\Psi(x,y) = \varphi(x) \cdot \phi(y)$$

Separation of variables leads to two differential equations for $\varphi(x)$ and $\phi(y)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{d^2x} + V(x) = E_x\varphi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{d^2y} = E_y\phi(y)$$

with

$$V(x,y) = \begin{cases} 0 & \text{for } 0 < \text{x} < \text{w} \\ \infty & \text{otherwise} \end{cases}$$

Solutions for $\phi(y)$ are plane waves:

$$\phi(y) = e^{iky}$$

with $\hbar k$ momentum in the y-direction and

$$E_y = \frac{\hbar^2 k^2}{2m}$$

Solutions for $\varphi(x)$ are those of the 1 dimensional infinite square well:

$$\varphi(x) = \sqrt{\frac{2}{w}} \sin\left(\frac{n\pi x}{w}\right)$$

$$E_x = \frac{n^2\pi^2\hbar^2}{2mw^2}$$

in which the factor $\sqrt{\frac{2}{w}}$ can be found by normalising the wave function.

10.b [1 p] The total energy is given by

$$E = E_x + E_y = \frac{n^2\pi^2\hbar^2}{2mw^2} + \frac{\hbar^2k^2}{2m}$$

The solutions are thus parabolic bands in the y direction that are offset from each other by an energy E_x that depends on the sub-band index, "n".

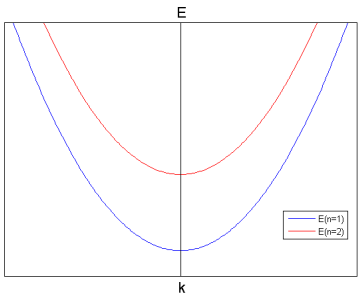


Figure 10.1

10.c [2 p] For electrons to behave one-dimensionally, a minimum requirement is that they occupy only the lowest subband and that they are not excited out of the lowest subband into the next one by thermal energy. This requires:

$$E(2,0) - E(1,0) > k_BT$$

$$\frac{3\pi^2\hbar^2}{2mw^2} > k_BT$$

$$\frac{3\pi^2\hbar^2}{2mk_BT} > w^2$$

$$w < \sqrt{\frac{3\pi^2\hbar^2}{2mk_BT}}$$

$\hbar c = 197eV \cdot nm$; $\mathrm{mc}^2 = 0.511$ MeV with $k_BT = \frac{1}{40}$eV at room temperature, so $k = \frac{1}{40}\frac{1}{300}\frac{eV}{K} = \frac{1}{1200}\frac{eV}{K}$

$$\sqrt{\frac{3\pi^2\hbar^2c^2}{2mc^2k_BT}} = \sqrt{\frac{3\pi^2(197eV\cdot nm)^2}{2\cdot 0.511MeV\frac{1}{1200}\frac{eV}{K}\cdot 0.05K}} = 165nm = w_max$$

So, the width should be smaller than 165 nm (ideally, w $\ll w_max$).

10.d [2 p] In order to ensure that you electrons are behaving one dimensionally, it is also important that you ensure that the Fermi energy stays only inside the first band when you fill up the bands of your wire. Otherwise, electrons have the freedom to move left and right in you wire in the x-direction. Asumming first you are at zero temperature, this requires

$$E_F < \frac{3\pi^2\hbar^2}{2mw^2}$$

For a one dimensional wire, the Fermi energy is given by:

$$E_F < \frac{\pi^2\hbar^2}{8m}n_{1d}^2$$

where n_{1d} is the number of electrons per unit length in the y direction.

$n_{1d} = n\cdot w$ gives

$$E_F < \frac{\pi^2\hbar^2}{8m}n^2w^2$$

Combining this with the upper limit on E_F

$$E_F < \frac{\pi^2\hbar^2}{8m}n^2w^2 < \frac{3\pi^2\hbar^2}{2mw^2}$$

$$n^2 < \frac{12}{w^4}$$

$$n < \frac{\sqrt{12}}{w^2} = 3.4\cdot 10^{16}\frac{\text{electrons}}{cm^2}$$

For those familiar with electron gasses in GaAs heterostructures, this is pretty easily achieved (typical $n \sim 10^{11}\frac{\text{electrons}}{cm^2}$). There will also be a correction from temperature, as we then want

$$E_F < \frac{3\pi^2\hbar^2}{2mw^2} - k_BT$$

But for 10 nm, the first term is ~3.6 meV ≈ 4.3 K ≪ 50mK and therefore the correction is small.

10.e [2 p] An electron in an electric field $\vec{E} = E_o\hat{z}$ moving with a velocity $\vec{v}$ will experience a magnetic field in it's reference frame given by:

$$\vec{B} = -\frac{1}{c^2}\vec{v} \times \vec{E}$$

In our wire, electrons can only flow in the y-direction: $\vec{v} = v\hat{y}$. This gives a magnetic field

$$\vec{B} = -\frac{1}{c^2}(vE_o)\hat{y} \times \hat{z} = -\frac{vE_0}{c^2}\hat{x}$$

10.f [2 p] In the reference frame of the moving electrons, there will be a Zeeman splitting of the electrons spin energy, given by:

$$E_{Zeeman} = -\vec{\mu} \cdot \vec{B}$$

$$\vec{\mu} = -\frac{g\mu_B\vec{S}}{\hbar}$$

$$\Rightarrow E_{Zeeman} = g\mu_B\vec{S} \cdot \vec{B}$$

Since the magnetic field points in the x-direction, the electron spin will align itself along the $+\hat{x}$ or $-\hat{x}$ direction with an energy difference given by:

$$\Delta E = \pm g\mu_B B = \pm g\mu_B\left(-\frac{vE_0}{c^2}\right)$$

$$v = \hbar k$$

$$\Delta E = \mp\frac{g\mu_B\hbar E_0}{c^2}k$$

To include this in our dispersion relation, we need to subtract this linear correction to the parabolas plotted in (b): adding or subtracting a straight line from a parabola results in a parabola whose origin is *not* at x = 0.

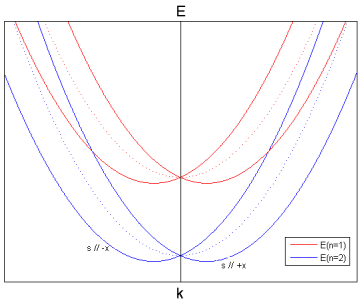


Figure 10.2

Note: lowest energy electrons acquire a finite momentum perpendicular to the electric field due to the Zeeman energy!

11. Ideal Qubits in canonical and microcanonical ensemble

Dr. Misha Titov
Radboud Universiteit Nijmegen
9 points

11.a [1 p] A simple way to find the free energy $F = -kT \ln Z_N$ is to compute first the partition function Z_N of the canonical ensemble

$$Z_N(T) = \sum_E e^{-\beta E},$$

where $\beta = 1/kT$ and the sum extends over all quantum states of the system. For non-interacting system of qubits we simply find $Z_N = Z_1^N$, where

$$Z_1 = 1 + e^{-\beta\varepsilon}.$$

Thus, we obtain the free energy as

$$F(N,T) = -kTN \ln\left(1 + e^{-\beta\varepsilon}\right).$$

In the limit of large temperate such that $kT \gg \varepsilon$ we find that free energy of the system is given $F = kTN \ln 2$. This corresponds to a classical limit when the free energy per single qubit is proportional to temperature irrespective of the value of ε. In the opposite limit one finds $F = 0$ since all qubits are in the ground state.

11.b [1.5 p] The computation of entropy amounts to taking the derivative of the free energy with respect to the temperature. We find

$$S = kN \ln\left(1 + e^{-\beta\varepsilon}\right) + kN \frac{\varepsilon}{kT} \frac{1}{e^{\beta\varepsilon} + 1}.$$

In the limit $\varepsilon \to 0$ we, therefore, obtain $S = kN \ln 2$ that corresponds to the Boltzmann formula of a microcanonical ensemble. In the opposite limit of zero temperature we obtain an exponentially small value of S,

$$S|_{T\to 0} = kNe^{-\beta\varepsilon}\left(\frac{\varepsilon}{kT} - 1\right).$$

The latter is due to the fact that the system at zero temperature is at a single ground state.

11.c [1.5 p] The average energy of the system of qubits can be expressed as

$$E = -\frac{1}{Z_N}\frac{\partial}{\partial\beta}Z_N = -\frac{\partial}{\partial\beta}\ln Z_N = \frac{N\varepsilon}{1+e^{\beta\varepsilon}}.$$

In the high temperature limit, the average energy per qubit is given by $\langle E\rangle/N = \varepsilon/2$. This is the manifestation of the so-called equipartition theorem. Intuitively, the result is clear since in the high temperature limit the probability to find the qubit in the state ε_0 and ε_1 are equal, hence the mean energy of each qubit is given by $(\varepsilon_1+\varepsilon_0)/2 = \varepsilon/2$.

11.d [2.5 p] Similarly to the mean energy we may express the mean square of energy as

$$\langle E^2\rangle = \frac{1}{Z_N}\frac{\partial^2}{\partial\beta^2}Z_N.$$

For the variance we have

$$\mathrm{varE} = \langle \mathrm{E}^2\rangle - (\langle \mathrm{E}\rangle)^2 = \frac{1}{\mathrm{Z_N}}\frac{\partial^2}{\partial\beta^2}\mathrm{Z_N} - \left(\frac{1}{\mathrm{Z_N}}\frac{\partial}{\partial\beta}\mathrm{Z_N}\right)^2 = \frac{\partial^2}{\partial\beta^2}\ln \mathrm{Z_N}.$$

The heat capacitance can be written as

$$C_V = T\frac{\partial S}{\partial T} = T\frac{\partial^2}{\partial T^2}kT\ln Z_N = k\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z_N.$$

Thus, we find

$$\mathrm{varE} = \mathrm{kT^2C_V}.$$

From the expression for the entropy we find

$$C_V = kN\left(\frac{\varepsilon}{kT}\right)^2\frac{e^{\varepsilon/kT}}{(1+e^{\varepsilon/kT})^2}. \tag{11.1}$$

This function clearly decays in both limits $\varepsilon \ll kT$ and $\varepsilon \gg kT$, since the fluctuations of the total energy of the system are suppressed in both limiting cases. The result is sketched in Figure 11.1a. The behaviour of C_V at large temperatures is the consequence of the fact that the spectrum of qubit system is restricted from below and from above. Indeed the maximal energy of the system of qubits is given by $N\varepsilon$, that is also an unique quantum state much like the ground state. Thus, in both high-temperature and low temperature limits the number of possibilities to distribute the excess energy over the system is strongly limited.

11.e [1 p] The total number of quantum states in the system is given by $W = 2^N$. The entropy is, then, given by the Boltzmann formula as

$$S = k \ln W = kN \ln 2.$$

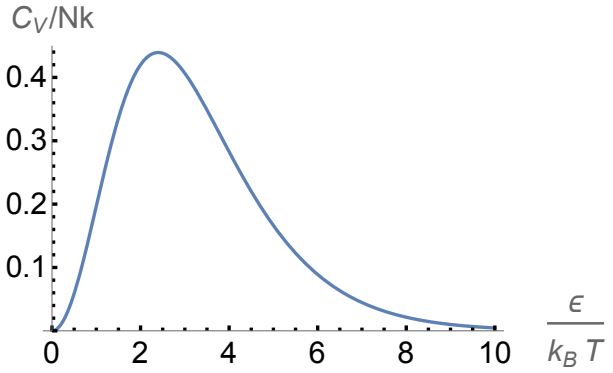
11.f [1.5 p] We have the relation

$$E = \frac{N\varepsilon}{1 + e^{\beta\varepsilon}}$$

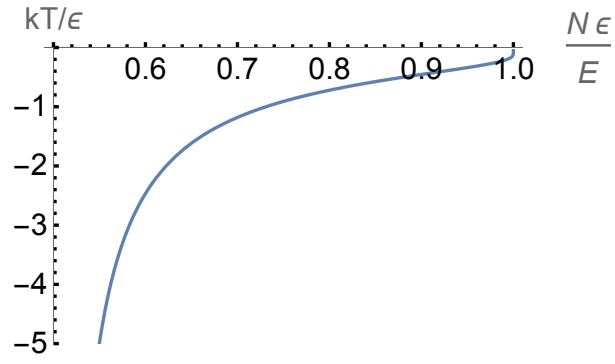
that can be used to find temperature for a fixed internal energy. This gives

$$kT = \frac{\varepsilon}{\ln\left(\frac{\varepsilon N}{E} - 1\right)}. \tag{11.2}$$

Note that the logarithm is negative for $N\varepsilon/2 < E < N\varepsilon$. The limit $E = N\varepsilon$ gives the maximal possible energy of the system that is characterised by a unique quantum state. For $E > N\varepsilon/2$ the number of available quantum states is decreasing with energy hence the negative temperature. The concept of negative temperature is well defined in the microcanonical ensemble even though it requires a rather exotic behaviour of the system such that the number of possibilities to distribute energy E over available microstates is decreasing with E. The result for temperature is sketched in Figure 11.1b.



(a) The dependence of C_V on the parameter ε/kT as given by Equation 11.1.



(b) The dependence of temperature on the total energy E as given by Equation 11.2.

Figure 11.1

12. Feynman-Hellmann theorem and the hydrogen atom

Prof. dr. Peter van der Straten
Universiteit Utrecht
10 points

12.a [2 p]

$$
\begin{aligned}
E_n(q) &= \int dr u_n^*(r,q)\mathcal{H}(\mathrm{r,q})\mathrm{u_n(r,q)} \\
\frac{\mathrm{d}E_n(q)}{\mathrm{d}q} &= \int dr \frac{\mathrm{d}u_n^*(r,q)}{\mathrm{d}q}\mathcal{H}(\mathrm{r,q})\mathrm{u_n(r,q)} + \int \mathrm{dru_n^*(r,q)}\frac{\mathrm{d}\mathcal{H}(\mathrm{r,q})}{\mathrm{dq}}\mathrm{u_n(r,q)} \\
&\quad + \int dr u_n^*(r,q)\mathcal{H}(\mathrm{r,q})\frac{\mathrm{du_n(r,q)}}{\mathrm{dq}} \\
&= E_n \int dr \frac{\mathrm{d}u_n^*(r,q)}{\mathrm{d}q}u_n(r,q) + \int dr u_n^*(r,q)\frac{\mathrm{d}\mathcal{H}(\mathrm{r,q})}{\mathrm{d}q}u_n(r,q) \\
&\quad + E_n \int dr u_n^*(r,q)\frac{\mathrm{d}u_n(r,q)}{\mathrm{d}q} \\
&= E_n \frac{\mathrm{d}}{\mathrm{d}q}\int dr u_n^* u_n + \int dr u_n^*(r,q)\frac{\mathrm{d}\mathcal{H}(\mathrm{r,q})}{\mathrm{d}q}u_n(r,q) \\
&= E_n \frac{\mathrm{d}}{\mathrm{d}q}1 + \int dr u_n^*(r,q)\frac{\mathrm{d}\mathcal{H}(\mathrm{r,q})}{\mathrm{d}q}u_n(r,q) \\
&= \int dr u_n^*(r,q)\frac{\mathrm{d}\mathcal{H}(\mathrm{r,q})}{\mathrm{d}q}u_n(r,q)
\end{aligned}
$$

12.b [1.5 p] Choose:
$q = Z$

$$
\begin{aligned}
\frac{\mathrm{d}E_n}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}\Big(-\frac{R_\infty Z^2}{n^2}\Big) = -\frac{2R_\infty Z}{n^2} \\
\frac{\mathrm{d}H}{\mathrm{d}Z} &= \frac{\mathrm{d}}{\mathrm{d}Z}\Big(-\frac{Ze^2}{4\pi\varepsilon_0 r}\Big) = -\frac{e^2}{4\pi\varepsilon_0 r}
\end{aligned}
$$

Thus

$$
\Big\langle -\frac{e^2}{4\pi\varepsilon_0 r}\Big\rangle = -\frac{e^2}{4\pi\varepsilon_0}\Big\langle\frac{1}{r}\Big\rangle = -\frac{2R_\infty Z}{n^2}
$$

12.c [1.5 p] Choose:
$q = \ell$
$n = n_r + \ell + 1$
$\Rightarrow \mathrm{d}\ell = \mathrm{d}n$

$$\frac{\mathrm{d}E_n}{\mathrm{d}\ell} = \frac{\mathrm{d}E_n}{\mathrm{d}n} = \frac{2R_\infty Z^2}{n^3}$$
$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\ell} = \frac{\ell + 1 + \ell}{2mr^2} = \frac{(\ell + \frac{1}{2})\hbar^2}{mr^2}$$
$$\Rightarrow \frac{(\ell+1)\hbar^2}{m}\langle\frac{1}{r^2}\rangle = \frac{2R_\infty Z}{n^3}$$

12.d [2 p]

$$\langle F(r)\rangle = \langle\frac{\mathrm{d}V_{int}(r)}{\mathrm{d}r}\rangle$$

Since V_{int} is an Hermitian operator, one can use the Feynman-Hellman theorem, but substitute H by V_{int}. Thus

$$\langle F(r)\rangle = \frac{\partial}{\partial r}\langle V_{int}\rangle = 0$$

Since $\langle V_{int}\rangle$ is independent of r.

12.e [1 p]

$$\frac{\mathrm{d}V_{int}}{\mathrm{d}r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2} - \frac{l(l+1)\hbar^2}{mr^3}$$

12.f [2 p] For ℓ=0 we have

$$u_n(0) = rR_{n\ell}(0) \propto cr$$

and thus

$$\left\langle\frac{1}{r^3}\right\rangle = \int_0^\infty c^2\frac{1}{r}\mathrm{d}r$$

which diverges for $r = 0$ and thus does not exist for ℓ=0.