# PLANCKS 19

Preliminaries, Italy
22 Feb 2019

# Exercises

PLANCKS 2019
Odense

{iaps}

AISF
associazione italiana studenti di fisica

## Physics League Across Numerous Countries for Kick-ass Students

# Italy Preliminaries

Exercises

*22nd February 2019*

## Contents



## Introduction

Dear contestants,
It is a pleasure to welcome you into the PLANCKS 2019 Italian Preliminaries! This booklet contains the exercises you will have to cope with in order to get access to the PLANCKS 2019 Finals in Odense, Denmark. You will face six different problems and are required to follow the instructions given in the next paragraphs.
Have fun and good luck!

# 1 Instructions

## 1.1 How it works

- The contest consists of 6 exercises each worth 20 points. Subdivision of points are indicated in the exercises.
- The test duration is for a total time of 2 hours and 30 minutes.
- When a problem is unclear, a participant can ask, via the crew, for a clarification from the organizing committee. The committee will respond to this request.
  If this response is relevant to all teams, we will provide this information to the other teams.
- In situations to which no rule applies, the organization decides.
- The organization has the right to disqualify teams for misbehaviour or breaking the rules.

## 1.2 What do you need

- You are only allowed to use a Italian/English dictionary and a non scientific calculator. **The use of hardware (including phones, tablets etc.) is not approved, with exceptions of simple watches and medical equipment**.
- No books or other sources, except for this exercise booklet and a dictionary are to be consulted during the competition.

## 1.3 What you are required to do

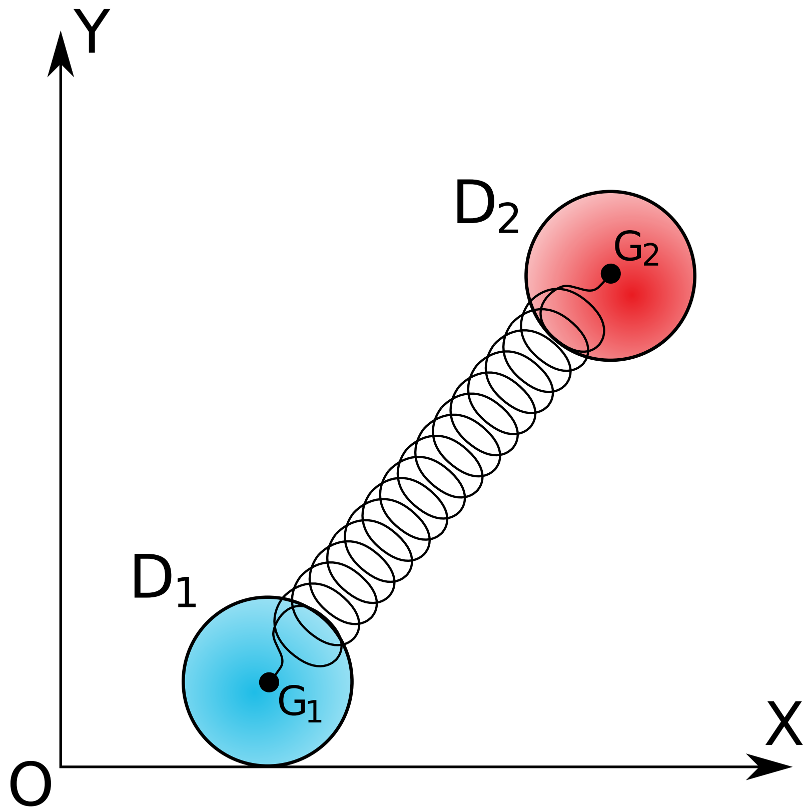
- The language used in the preliminary and international competition is English.
- Solutions **and** procedure must be included in the answer paper.
- All exercises must be handed in separately. **Please use a separate sheet for each problem.**
- Respect all the given instructions.
- Enjoy and have a great physics time!

# 2 Problems

## 2.1 Mechanics

### Two disks and an ideal spring

In a vertical plane $(O, X, Y)$, consider the motion of two identical rigid disks $D_1$ and $D_2$ of radius $R$ and mass $M$. Disk $D_1$ rolls without slipping on the horizontal axis $X$. The disks are subjected to the gravity force, with $g = 9.81 \frac{m}{s}$, and an ideal spring of constant $k > 0$ is applied between the centers of mass $G_1$ e $G_2$ of the two disks.




1. Determine the Lagrange function and equations.
   ***[12 points]***

2. Find the conservation laws.
   ***[4 points]***

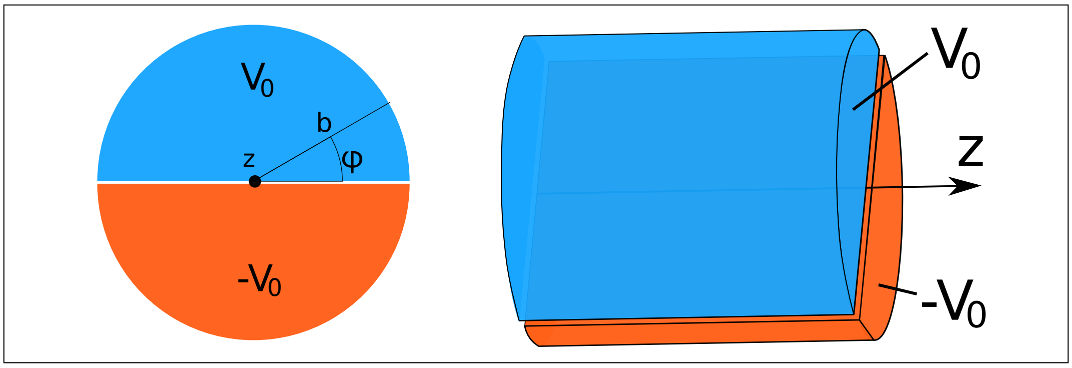
3. Can disk $D_2$ remain at the same height? Justify your answer.
   ***[4 points]***

## 2.2 Electromagnetism

### Potential generated by two infinite length half cylinders

Consider an infinite cylinder with radius $b$. The cylinder is made of two isolated half-cylinders as depicted in the figure (imagine to cut the cylinder along its length).
The upper half-cylinder has a potential $+V_0$, while the potential of the lower half-cylinder is $-V_0$.

1. Compute the potential inside the cylinder.
   ***[20 points]***




Useful formula:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.1}$$

## 2.3 Quantum Mechanics

### Einstein coefficients and harmonic oscillator

In a seminal paper in 1916 Einstein showed how Planck distribution for equilibrium radiation at temperature $T$:

$$u(\omega)d\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\exp(\frac{\hbar\omega}{kT})-1}d\omega; \qquad \omega = 2\pi\nu\,, \tag{2.2}$$

could be deduced introducing the coefficients of spontaneous and induced emission, $A_{a\to b}$, $B_{a\to b}$, and the absorption coefficient $B_{b\to a}$ for a two level system, with $E_a > E_b$. In this exercise, we will consider how Einstein procedure can be applied to a one dimensional harmonic oscillator in thermal equilibrium with the radiation, i.e. the original system studied by Planck. As in the original Einstein paper we assume that the asymptotic behaviors for $u(\omega)$ are known:

$$\begin{aligned} T\to\infty: \quad & u(\omega)\simeq \frac{\omega^2}{\pi^2 c^3}kT\,; \qquad \text{Rayleigh-Jeans law} \\ T\to 0: \quad & u(\omega)\simeq K\exp(-\tfrac{\hbar\omega}{kT})\,; \quad \text{Wien law} \end{aligned} \tag{2.3}$$

We assume that the oscillator has a proper (angular) frequency $\omega_0$ and that in thermal equilibrium the Boltzmann distribution holds:

$$P_n^{eq} = P_0^{eq} e^{-n\hbar\omega_0/kT}\,; \quad n = 0, 1, 2\ldots \tag{2.4}$$

$n$ is the usual label for oscillator energy levels. For simplicity we will suppose that only transitions with $\Delta n = \pm 1$ are allowed.

1. Introduce the appropriate coefficients $A, B$ and write the rate equations and the relative equilibrium conditions for the probability $P_n$, defined as the probability of finding the oscillator in the $n$-th level.
   ***[6 points]***
2. Find the relation between absorption and induced emission coefficients. Hint: Use the fact that for $T\to\infty$ $u(\omega)$ must grow indefinitely.
   ***[6 points]***
3. Use the Wien limit form for $u(\omega)$ to obtain the general form of Planck distribution and fix $K$ by Rayleigh-Jeans limit.
   ***[8 points]***

*Prof. Giampiero Paffuti*
*INFN - UniPi, Pisa*

## 2.4 Particle Physics

### $\pi^+ \to \mu^+\nu_\mu$ process

A collimated beam of charged pions ($\pi^+$) is prepared with $N_\pi^{(0)} = 10^6$ particles with energy $E_\pi = 10$ GeV.
Charged pions decay as $\pi^+ \to \mu^+\nu_\mu$. After a travel length $L = 100$ m, a shield absorbs all pions and only muons are left through.

1. Compute how many muons $N_\mu^{(L)}$ go through the shield.
   ***[5 points]***

2. Compute the momentum $p^*$ of the decay products and the energy of the muon $E_\mu^*$ in the center-of-mass reference frame (CMRF).
   ***[5 points]***

3. Compute the minimum and maximum energy $E_\mu^{min}$, $E_\mu^{max}$ of the muons in the LAB reference frame (LABRF).
   ***[5 points]***

4. Prove that the distribution of the muon energy $E_\mu$ is *flat* in the interval $[E_\mu^{min}\ ;\ E_\mu^{max}]$ [suggestion: call $\theta^*$ the decay angle of the muon in the CMRF, with respect to the flying direction of the pion: how is $\cos\theta^*$ distributed?]
   ***[5 points]***

Some notes:

- Energies, momenta, and masses are expressed in MeV or GeV — i.e. for momenta $p$ and masses $m$ we actually give/ask for $pc$ and $mc^2$.
- Properties of the pion: mass $m_\pi = 139.6$ MeV ; average lifetime: $\tau_\pi = 2.6 \cdot 10^{-8}$ s ; spin $= 0$
- Properties of the muon: mass $m_\mu = 105.7$ MeV ; average lifetime: $\tau_\mu = 2.2 \cdot 10^{-6}$ s ; spin $= 1/2$

## 2.5 Cosmology

# Rotation Curve of a Galaxy

The rotation curve of a galaxy is the variation of its circular orbit velocity $v(r)$ as a function of the distance from the galactic center, $r$. In case of spherical symmetry and Newtonian gravity, the rotational velocity can be easily related to $M(r)$, the mass of the galaxy contained within $r$.
A contribution to the orbital velocity is certainly due to the stellar disc, whose surface density can be described by

$$\Sigma_D(r) = \Sigma_{D,0} \exp(-r/R_D) \tag{2.5}$$

where $R_D$ is a characteristic length and $\Sigma_{D,0}$ the surface density at $R_D$.
An additional component due to a spherical dark matter halo can be present. Suppose that its volume mass density follows $\rho_H(r) = \rho_0 \frac{r_0^2}{r^2+r_0^2}$, where $r_0$ is a typical scale-length and $\rho$ is the density at $r_0$.

1. Write the relation between $v(r)$ and $M(r)$.
   ***[3 points]***
2. Derive the relation between $\rho(r)$ and $v(r)$
   ***[3 points]***
3. Compute the mass of the stellar disc encompassed in a radius $r$, $M_D(r)$, and the corresponding rotational velocity $v_D(r)$.
   ***[4 points]***
4. Compute the mass of the dark halo encompassed in a radius $r$, $M_H(r)$, and the corresponding rotational velocity $v_H(r)$.
   ***[5 points]***
5. Draw qualitatively the trend of $v_D(r)$ and $v_H(r)$, and outline the predictions at high radii.
   ***[5 points]***

## 2.6 Solid State Physics

# Heat Capacity in a 1D monoatomic crystal model

A one-dimensional monoatomic crystal of length L can be represented as a linear array of $(N+1)$ identical masses $M$, with spacing $a << L$ and interconnected by forces (springs), which are represented in their harmonic approximation with force constant $(\beta)$.
With the assumption that any oscillator is connected only to its nearest neighbours, the dynamics of the crystal is given in terms of normal modes, which, when treated quantum mechanically, are identified with phonons.
Given the dispersion relation linking the wavenumber k of the normal mode and the characteristic frequency $(\omega)$:

$$\omega = \omega_M \left| \sin \frac{k_a}{2} \right|, \qquad \omega_M = 2\sqrt{\frac{\beta}{M}} \tag{2.6}$$

the thermodynamic properties of such a system can be exactly derived.
The goal of this problem is to derive the integral expression of the heat capacity from this model in the following scenarios.

1. Demonstrate that for very high temperatures (i.e. for $T \to \infty$), the heat capacity asymptotically converges to the classical (one-dimensional) Dulong-Petit law.
   ***[10 points]***

2. In the high temperature limit, i.e. when $T >> \frac{\hbar\omega_M}{k_B}$, the approximation of the thermal capacity allows an expression of the Einstein temperature $T_E$, within the approximation introduced by Einstein in 1907, which assume that all the N oscillators vibrate at the same frequency $\omega_E = \frac{k_B T_E}{\hbar}$.
   ***[10 points]***

N.B. Let consider the following series expansion, for $s \to 0$, $\frac{e^s}{(e^s-1)^2}s^2 \approx 1 - \frac{s^2}{12}$.