A cart

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December 27, 2024

Problem

The cart represented in Figure 1 has been assembled by connecting together four identical cylinders, with mass m and radius R, using a rigid structure with negligible mass. The cylinders rotate without slipping on the horizontal planes they are in contact with.

Each upper cylinder is connected with the lower one with a drive belt, adherent to two pulleys with radius r_1 and r_2. Friction is negligible.

The lower plane do not move, while the upper one moves horizontally with velocity $v > 0$.

1. Evaluate the ratio $\gamma \equiv \omega_1/\omega_2$ between the angular velocities of upper and lower cylinders.

2. Evaluate and plot the ratio between the velocity of the center of mass of the cart v_{cm} and the velocity of the upper plane, as a function of. Is it possible to have $v_{cm} = v$? Is it possible that $v_{cm} > v$? Is it possible that the cart moves in the opposite direction of the plane? What happens when $\gamma \to -1$?

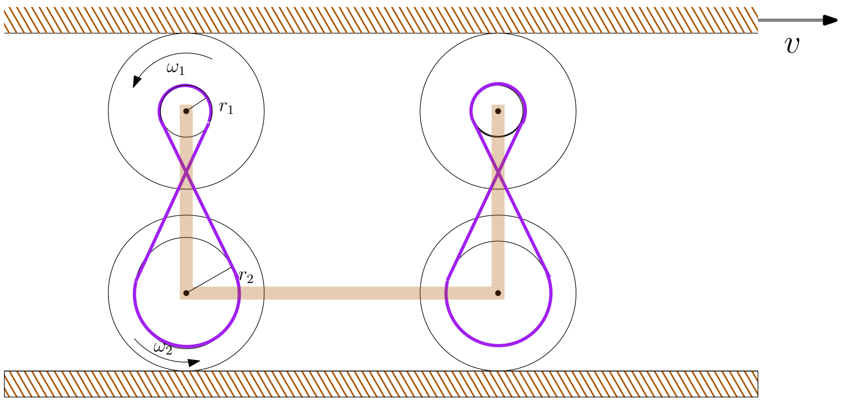

Figure 1: Schematic representation of the cart.

3. Suppose that the upper plane is not moving initially. Evaluate the work needed to change its velocity to v. Explain what happens when $\gamma \to -1$.

Solution

Question 1

In the reference frame of the cart, we see that the velocity of the drive belt is tangent to it in each point and with the same speed. Evaluating the speed on the upper and lower pulley, we see that

$$\omega_1 r_1 = -\omega_2 r_2$$

so it follows that

$$\gamma = \frac{\omega_1}{\omega_2} = -\frac{r_2}{r_1} < 0$$

Question 2

Using the no slipping condition on the lower cylinder we find

$$v_{cm} = -R\omega_2$$

while on the upper one the same condition gives

$$v_{cm} = v + R\omega_1$$

From these two relations we get

$$\frac{v_{cm} - v}{v_{cm}} = -\gamma$$

and finally

$$\frac{v_{cm}}{v} = \frac{1}{1+\gamma}$$

The ratio v_{cm}/v is represented in Figure 2 as a function of γ.

- We see that when $\gamma \to 0$ then $v_{cm} \to v$. We approach this condition when $r_1 \gg r_2$: in this case the upper cylinder ha an angular velocity $\omega_1 \to 0$, end for the no slipping condition the cart must move with a velocity very similar to the upper plane's one.
- We get $v_{cm} > v$ when $-1 < \gamma < 0$.
- The cart moves in the opposite direction of the plane (i.e. $v_{cm} < 0$) when $\gamma < -1$.
- The ratio v_{cm}/v grows without limit when $\gamma \to -1$: this correspond to $r_1 \to r_2$. In this case $\omega_1 \to -\omega_2$ but it must also be true that

  $$v + R\omega_1 = -R\omega_2$$

  For a given velocity v this is possible only if the angular velocities (and v_{cm}) increases without limit.

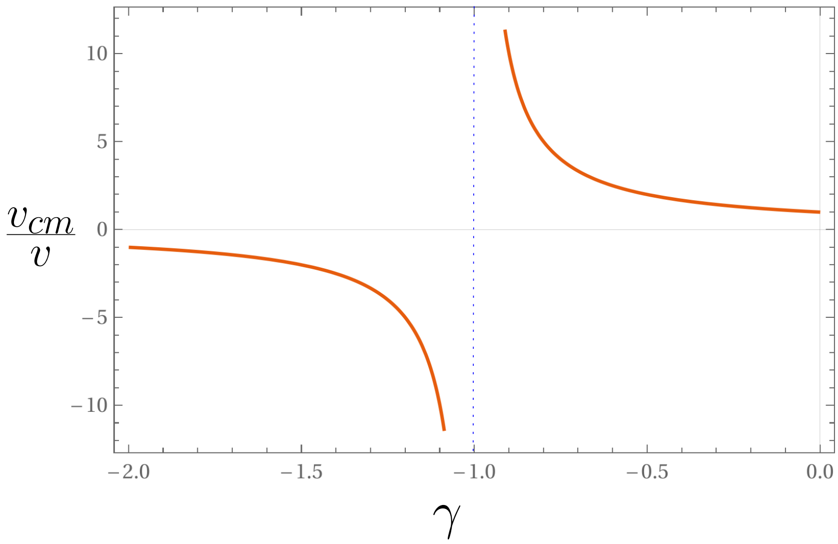

Figure 2: The ratio v_{cm}/v between the velocity of the cart and the velocity of the upper plane, as a function of $\gamma < 0$.

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The kinetic energy of the cart is given by

$$\begin{aligned} K &= 2 \times \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega_1^2 + 2 \times \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega_2^2 + \frac{1}{2}4mv_{cm}^2 \\ &= \left(\frac{1}{2}mR^2\right)\left(\omega_1^2 + \omega_2^2\right) + 2mv_{cm}^2 \\ &= \frac{5+\gamma^2}{(1+\gamma)^2}\frac{1}{2}mv^2 \end{aligned}$$

and this is also the work that must be done to change the plane velocity from 0 to v.

When $\gamma \to -1$ this quantity increases without limit: in other words if $\gamma \to -1$ in order to allow a non zero velocity for the cart a larger and larger work will be needed. When $\gamma = -1$ it will not be possible to move the cart.

An Antenna-Fed Cavity

An one-dimensional cavity model is composed by two conducting planes at $x = \pm a/2$ and an infinitely thin antenna at $x = 0$ with a current density $\mathbf{J}(x, y, z, t) = K(t)\,\delta(x)\,\hat{\mathbf{y}}$ with $K(t) = \mathrm{Re}[K_0\,\mathrm{e}^{-i\omega t}]$, driven by an ideal current generator.

Assume in the first instance that the planes are made by perfect conductors.

a) Find the expression for the electromagnetic (EM) field and discuss the presence of resonances.

b) Determine the instantaneous power exchanged between the EM field and the current generator.

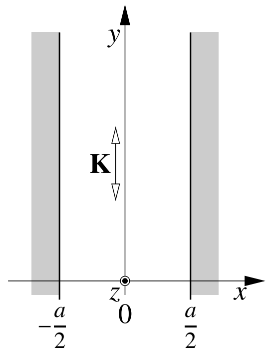

Figure 1

Now assume that the conductors have finite conductivity, so that they are characterized by a non-zero skin depth δ.

c) Find again the EM field and the exchanged power (first discuss the case of resonant modes and then the general expression).

Solution

a) Since the antenna current is along $\hat{\mathbf{y}}$ direction, the electric and magnetic field have the $E_y = E_y(x,t)$ and $B_z = B_z(x,t)$ components, respectively. The general solutions are (in complex notation) $E_y(x,t) = E_y(x)\,\mathrm{e}^{-i\omega t}$ and $B_z(x,t) = B_z(x)\,\mathrm{e}^{-i\omega t}$. Because of the symmetry of the problem and the continuity of E_y at $x = 0$ (as a consequence of the Faraday-Maxwell equation $c\partial_x E_y = -\partial_t B_z$) , we can take E_y as an even function and restrict ourselves to the $0 < x < a/2$ region. The F-M equation implies that B_z is odd, which is consistent with the discontinuity of B_z at $x = 0$ because of the surface current. The general solution for the fields can be written as

$$E_y(x) = E_+\mathrm{e}^{ikx} + E_-\mathrm{e}^{-ikx} \,, \tag{1}$$

$$B_z(x) = E_+\mathrm{e}^{ikx} - E_-\mathrm{e}^{-ikx} \,, \tag{2}$$

where $k = \omega/c$. The jump condition for B_z, which is obtained by integrating the Ampére-Maxwell equation $c\partial_x B_z = -4\pi J_y - \partial_t E_y$ across the discontinuity, is

$$B_z(0^+,t) - B_z(0^-,t) = \frac{4\pi}{c}K(t) \,. \tag{3}$$

Up to now all the above considerations are independent on the conductivity of the cavity walls. For the perfect conductors case, $E_y(\pm a/2, t) \equiv 0$. Thus the complete boundary conditions for the EM field are

$$E_+\mathrm{e}^{ika/2} + E_-\mathrm{e}^{-ika/2} = 0 \,, \tag{4}$$

$$E_+ - E_- = \tfrac{2\pi}{c}K_0 \,, \tag{5}$$

with solutions for E_+ and E_-

$$E_\pm = \pm\frac{\pi}{c}K_0\frac{\mathrm{e}^{\mp ika/2}}{\cos(ka/2)} \,. \tag{6}$$

The singularities for $ka/2 = (n + 1/2)\,\pi$, with $n \geq 0$ an integer, correspond to the resonant modes for which the cavity length is an odd-integer number of wavelengths i.e. $\lambda_n = 2\pi/k_n = a/(n + 1/2)$. These modes can exist inside the cavity even in the absence of the antenna.

b)

In steady state, the energy flow through the antenna must be equal to the work done per unit time (i.e. the power) by the electric field on the current. Assuming K_0 as real and taking the real part of the fields, the power per unit surface is

$$P_\mathrm{s}(t) = K(t)\,E_y(0,t) = (K_0\cos\omega t)\left[-\tfrac{2\pi}{c}K_0\tan\left(\tfrac{ka}{2}\right)\,\sin\omega t\right] = -\tfrac{\pi}{c}K_0^2\tan\left(\tfrac{ka}{2}\right)\,\sin 2\omega t \,. \tag{7}$$

The same results can be also obtained by evaluating the Poynting vector at $x = 0^\pm$. The power vanishes for modes with $ka/2 = n\pi$ for which $E_y(0,t) = 0$

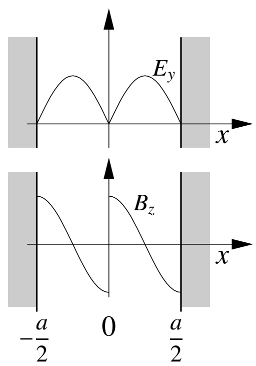

Figure 2

(Fig.2 shows a sketch of the lowest frequency mode). Out of resonance, the energy flow oscillates at 2ω, i.e. there is a periodic exhange of energy between the field and the generator.
c) For $x > a/2$, the EM fields are evanescent as

$$E_y(x) = E_s \mathrm{e}^{-(1-\mathrm{i})(x-a/2)/\delta} \,, \qquad B_z(x) = \frac{\mathrm{i}+1}{k\delta} E_s \mathrm{e}^{-(1-\mathrm{i})(x-a/2)/\delta} \qquad (x > a/2) \,, \tag{8}$$

where $E_s = E_y(x = a/2)$. The boundary conditions are now

$$E_+ \mathrm{e}^{\mathrm{i}ka/2} + E_- \mathrm{e}^{-\mathrm{i}ka/2} = E_s \,, \tag{9}$$
$$E_+ \mathrm{e}^{\mathrm{i}ka/2} - E_- \mathrm{e}^{-\mathrm{i}ka/2} = \tfrac{\mathrm{i}+1}{k\delta} E_s \,, \tag{10}$$
$$E_+ - E_- = \tfrac{2\pi}{c} K_0 \,. \tag{11}$$

For the normal modes with $ka = \pi(2n+1)$, using $\mathrm{e}^{\pm \mathrm{i}\pi(n+1/2)} = \pm(-1)^n \mathrm{i}$ Eqs.(9, 10, 11) become

$$(-1)^n i E_+ - (-1)^n \mathrm{i} E_- = E_s \,, \tag{12}$$
$$(-1)^n i E_+ + (-1)^n \mathrm{i} E_- = \tfrac{\mathrm{i}+1}{k\delta} E_s \,, \tag{13}$$
$$E_+ - E_- = \tfrac{2\pi}{c} K_0 \,, \tag{14}$$

with solution

$$E_\pm = \pm \frac{\pi}{c} K_0 \left(1 \pm \frac{\mathrm{i}+1}{k\delta} \right) \,. \tag{15}$$

The average power (per unit surface) provided by the antenna is

$$\langle P_\mathrm{s}(t) \rangle = \frac{1}{2} \mathrm{Re}(K_0 E_0^*) = -\frac{\pi}{c} \frac{K_0^2}{k\delta} = -\frac{1}{2n+1} \frac{K_0^2}{c} \frac{a}{\delta} \,. \tag{16}$$

For a generic value of $\omega = kc$, Eqs.(9, 10, 11) have the solution

$$E_\pm = \pm \frac{2\pi}{c\Delta} K_0 \left(\frac{1+\mathrm{i}}{k\delta} \mp 1 \right) \mathrm{e}^{\mp \mathrm{i}ka/2} \,, \qquad E_s = -\frac{4\pi K_0}{c\Delta} \,. \tag{17}$$

where

$$\Delta = -\frac{2(1+\mathrm{i})}{k\delta} \cos(ka/2) + 2\mathrm{i} \sin(ka/2) \,. \tag{18}$$

Thus

$$\begin{aligned} \langle P_\mathrm{s}(t) \rangle &= \tfrac{1}{2} \mathrm{Re}(K_0 E_0^*) = \tfrac{\pi}{c} K_0^2 \, \mathrm{Im} \left[\tfrac{\sin(ka/2) + k\delta \frac{1+\mathrm{i}}{2} \cos(ka/2)}{\cos(ka/2) - k\delta \frac{1+\mathrm{i}}{2} \sin(ka/2)} \right] \\ &= -\tfrac{\pi}{2c} K_0^2 \tfrac{k\delta}{[\cos(ka/2) + (k\delta/2)\sin(ka/2)]^2 + (k\delta/2)^2 \sin^2(ka/2)} \,, \end{aligned} \tag{19}$$

which is non-vanishing for all modes and equal the energy dissipated inside the conducting walls.

Quantum Mechanics

Consider three particles of spin 1/2 in the state

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\Big) ,$$

where $|\uparrow\uparrow\downarrow\rangle$ is the eigenstate of the z-components S_{1z}, S_{2z}, S_{3z} of the spin of the three particles corresponding to the eigenvalues $+\hbar/2$, $+\hbar/2$, $-\hbar/2$ respectively, and similarly for $|\uparrow\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\uparrow\rangle$.

a) Find the possible results and corresponding probabilities of the measurement of S^2 and S_z, where $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ is the total spin of the three particles **(5 points)**.

b) Find the possible results and corresponding probabilities of the independent measurements of S_{1z}, $S_{1z} + S_{2z}$ and S_{1x} **(5 points)**.

Suppose that a measurement of the x-component S_{1x} of the spin of the first particle is carried out, leading to the result $S_{1x} = +\hbar/2$.

c) Find the state of the particles after the measurement and the possible results and corresponding probabilities of the independent measurements of S^2 and S_z **(10 points)**,

Solution

a) The addition of angular momenta for three particles of spin 1/2 can be performed by first adding the spin of two particles, $\vec{S}_{12} = \vec{S}_1 + \vec{S}_2$, and then the spin of the third particle, $\vec{S} = \vec{S}_{12} + \vec{S}_3 = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$. In terms of spin quantum numbers, we have:

$$s_1 = 1/2\ ,\ s_2 = 1/2 \longrightarrow s_{12} = 0\,,\,1\ ;\ \Longrightarrow \begin{array}{l} s_{12} = 1\ ,\ s_3 = 1/2 \longrightarrow s = 1/2\,,\,3/2 \\ s_{12} = 0\ ,\ s_3 = 1/2 \longrightarrow s = 1/2 \end{array} \tag{1}$$

Therefore, in the basis of eigenstates of the total spin S^2 and S_z, we have one quadruplet of spin $s = 3/2$ and two doublets of spin $s = 1/2$.

The states of maximum spin $s = 3/2$ are fully symmetric with respect to the exchange of the three particles. Since $|\psi_0\rangle$ is fully symmetric and it is an eigenstates of total S_z with $s_z = s_{1z} + s_{2z} + s_{3z} = 1/2$, we can conclude that

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\Big) = |3/2, 1/2\rangle\ , \tag{2}$$

where we denote with $|s, s_z\rangle$ the eigenstates of the total spin operators S^2 and S_z. We can arrive to the same result as in Eq. (2) by explicitly computing the Clebsh-Gordan coefficients in the standard way, i.e. starting form the state of maximum angular momentum,

$$|3/2, 3/2\rangle = |\uparrow\uparrow\uparrow\rangle \tag{3}$$

and applying to both sides of the equation the ladder operator S_-:

$$\begin{aligned} S_-|3/2,3/2\rangle &= \hbar\sqrt{3}\,|3/2,1/2\rangle = (S_{1-}+S_{2-}+S_{3-})\,|\uparrow\uparrow\uparrow\rangle \\ &= \hbar\,(\,|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\,)\ , \end{aligned} \tag{4}$$

which leads again to Eq. (2).

Being $|\psi_0\rangle$ an eigenstate of S^2 and S_z, the measurements of the two observables lead to the corresponding eigenvalues with unitary probability, namely

$$\boxed{P(s=3/2)=1 \quad , \quad P(s_z=1/2)=1\ ,} \ . \tag{5}$$

where, as well known, the quantum numbers $s = 3/2$ and $s_z = 1/2$ correspond to $S^2 = 15/4\,\hbar^2$ and $S_z = 1/2\,\hbar$.

b) From Eq. (2) it follows that a measurement of S_{1z}, S_{2z}, S_{3z} in the state $|\psi_0\rangle$ leads to the following results and corresponding probabilities p_i:

$$\begin{aligned} s_{1z} &= +\frac{1}{2}\ ,\ s_{2z} = +\frac{1}{2}\ ,\ s_{3z} = -\frac{1}{2} \longrightarrow p_1 = \frac{1}{3} \\ s_{1z} &= +\frac{1}{2}\ ,\ s_{2z} = -\frac{1}{2}\ ,\ s_{3z} = +\frac{1}{2} \longrightarrow p_2 = \frac{1}{3} \\ s_{1z} &= -\frac{1}{2}\ ,\ s_{2z} = +\frac{1}{2}\ ,\ s_{3z} = +\frac{1}{2} \longrightarrow p_3 = \frac{1}{3} \end{aligned} \tag{6}$$

The following probabilities for S_{1z} and $S_{1z} + S_{2z}$ are then readily derived:

$$\boxed{P(s_{1z}=1/2) = p_1 + p_2 = \frac{2}{3} \quad , \quad P(s_{1z}=-1/2) = p_3 = \frac{1}{3}}\ , \tag{7}$$

and

$$\boxed{P(s_{1z}+s_{2z}=1) = p_1 = \frac{1}{3} \quad , \quad P(s_{1z}+s_{2z}=0) = p_2 + p_3 = \frac{2}{3}}\ . \tag{8}$$

In order to determine results and probabilities for the measurement of S_{1x}, we express the state $|\psi_0\rangle$ in terms of the eigenstates $|+\rangle$ and $|-\rangle$ of S_{1x}, corresponding to the eigenvalues $\pm\,\hbar/2$. These states are related to the eigenstates of S_{1z} by

$$\begin{cases} |+\rangle = \dfrac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \\ |-\rangle = \dfrac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{cases} \Longrightarrow \begin{cases} |\uparrow\rangle = \dfrac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \\ |\downarrow\rangle = \dfrac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right) \end{cases} . \tag{9}$$

The state $|\psi_0\rangle$ can be then written as

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{3}}\Big[\,|\uparrow\rangle\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + |\downarrow\rangle\left(|\uparrow\uparrow\rangle\right)\Big] = \\ &= \frac{1}{\sqrt{6}}\Big[\left(|+\rangle + |-\rangle\right)\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) + \left(|+\rangle - |-\rangle\right)\left(|\uparrow\uparrow\rangle\right)\Big] = \\ &= \frac{1}{\sqrt{6}}|+\rangle\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle\right) + \frac{1}{\sqrt{6}}|-\rangle\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\uparrow\rangle\right) . \end{aligned} \tag{10}$$

From Eq. (10), the following results and probabilities for the measurement of S_{1x} are obtained:

$$\boxed{P(s_{1x}=1/2)=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2} \quad , \quad P(s_{1x}=-1/2)=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}} \ . \tag{11}$$

c) The state of the three particles after the measurement of S_{1x} with the result $s_{1x}=+1/2$ can be read from Eq. (10):

$$\begin{aligned} |\psi_0'\rangle &= \frac{1}{\sqrt{3}}|+\rangle\,\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle\big) = \\ &= \frac{1}{\sqrt{6}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\Big) \end{aligned} \tag{12}$$

We now want to express $|\psi_0'\rangle$ in terms of the eigenstates of S^2 and S_z. We first observe that using Eqs. (2) and (3), $|\psi_0'\rangle$ can be written as

$$|\psi_0'\rangle = \frac{1}{\sqrt{6}}|3/2,3/2\rangle + \frac{1}{\sqrt{2}}|3/2,1/2\rangle + \frac{1}{\sqrt{3}}|\varphi\rangle \tag{13}$$

where we have defined

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\Big(|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\Big) \ . \tag{14}$$

The state $|\varphi\rangle$ is clearly an eigenstate of S_z with $s_z=-1/2$. Therefore, expressed in terms of the eigenstates of S^2 and S_z, it must be a linear combination of the form

$$|\varphi\rangle = a_{3/2}\,|3/2,-1/2\rangle + a_{1/2}\,|1/2,-1/2\rangle \ . \tag{15}$$

where $a_{3/2}$ and $a_{1/2}$ are complex coefficients. The explicit expression of the eigenstate $|3/2,-1/2\rangle$ can be obtained from Eq. (2) by inverting the z-components of the spin, namely

$$|3/2,-1/2\rangle = \frac{1}{\sqrt{3}}\Big(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle\Big) \ . \tag{16}$$

Therefore, the coefficient $a_{3/2}$ can be readily computed:

$$a_{3/2} = \langle 3/2,-1/2|\varphi\rangle = \frac{1}{\sqrt{6}}\Big(\langle\downarrow\downarrow\uparrow| + \langle\downarrow\uparrow\downarrow| + \langle\uparrow\downarrow\downarrow|\Big)\Big(|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\Big) = \sqrt{\frac{2}{3}} \ . \tag{17}$$

The coefficient $a_{1/2}$ in Eq. (15) is defined only up to an arbitrary moltiplicative phase factor which defines, in turn, the phase of the state $|1/2,-1/2\rangle$. Choosing this factor in such a way that $a_{1/2}$ is real and positive, allows us to determine the coefficient from the normalization condition,

$$a_{1/2} = \sqrt{1-|a_{3/2}|^2} = \sqrt{\frac{1}{3}} \ . \tag{18}$$

By substituting Eqs. (17) and (18) in Eq. (15) and using the result in Eq. (13), we finally arrive at the expression of the state $|\psi_0'\rangle$ in terms of the eigenstates of S^2 and S_z,

$$|\psi_0'\rangle = \frac{1}{\sqrt{6}}|3/2,3/2\rangle + \frac{1}{\sqrt{2}}|3/2,1/2\rangle + \frac{\sqrt{2}}{3}|3/2,-1/2\rangle + \frac{1}{3}|1/2,-1/2\rangle \ . \tag{19}$$

This expression leads in turn to the following results and probabilities for the measurements of S^2 and S_z:

$$\boxed{P(s=3/2)=\frac{1}{6}+\frac{1}{2}+\frac{2}{9}=\frac{8}{9} \quad , \quad P(s=1/2)=\frac{1}{9}} \quad , \tag{20}$$

and

$$\boxed{P(s_z=3/2)=\frac{1}{6} \quad , \quad P(s_z=1/2)=\frac{1}{2} \quad , \quad P(s_z=-1/2)=\frac{2}{9}+\frac{1}{9}=\frac{1}{3}} \quad . \tag{21}$$

PrePLANCKS

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5 February 2025

Exercise

A one-dimensional chain is made by a sequence of N molecules having each a 2D ellipsoidal shape with minor and major axis of length b and a ($b < a$) respectively. Let us call x the direction of the 1D chain. Each molecule can be oriented in space in two directions, one in which a is oriented along x and the other in which b is oriented along x. Suppose that there is a force $\vec{F}$ applied to the N-esim molecule along the chain axis x. The energy of the system is given by the Hamiltonian

$$\mathcal{H} = -FL\left[\{l_i\}\right] = -F\sum_{i}^{N} l_i \tag{1}$$

where l_i can be either a or b.

(a) Compute the canonical partition function of the system and its average energy $\langle E \rangle$. Verify the accuracy of the solution by calculating the limits of $\langle E \rangle$ at low and high temperatures.

(b) Compute the equilibrium average length $\langle L \rangle$ of the chain as a function of the force strength F and temperature T. Furnish the explicit expression of $\langle \ell \rangle \equiv \langle L \rangle / N$ as a function of F/T and obtain the limiting case of large forces ($Fb/k_BT >> 1$);

(c) Show that in the weak forces regime ($Fa/k_BT << 1$) the chain satifies the relation

$$g(T)(\langle L \rangle - L_0) = F \tag{2}$$

and determine $g(T)$ and $L_0 \equiv \ell_0 N$;

(d) Compute the free energy of the system, $f(T, F)$ as a function of T and $\langle L \rangle$ in the weak forces regime.

Solution

(a) Since the Hamiltonian si the sum of N independent, i.e. $\mathcal{H} = \sum_i^N -Fl_i$ the partition function is simply

$$\begin{aligned} Z_N(F,T) &= \sum_{\{l_i\}} e^{-\beta\mathcal{H}[\{l_i\}]} \\ &= \sum_{\{l_i\}} e^{\beta\sum_i^N Fl_i} \\ &= \prod_{i=1}^{N} \left(\sum_{l_i=a,b} e^{\beta F l_i} \right) \\ &= \zeta^N \end{aligned} \tag{3}$$

with

$$\zeta(F,\beta) = \sum_{l_i=a,b} e^{\beta F l_i} = e^{\beta F a} + e^{\beta F b} \tag{4}$$

and $\beta \equiv \frac{1}{k_B T}$.
Hence

$$\langle E \rangle = -\frac{\partial Z_N(F,T)}{\partial \beta} = -NF\frac{ae^{\beta Fa} + be^{\beta Fb}}{e^{\beta Fa} + e^{\beta Fb}} \tag{5}$$

As expected, at very high temperatures ($\beta \to 0$) $\langle E \rangle = -NF(a+b)/2$ i.e. the two positions are equiprobable. On the other hand, at very low temperature ($\beta \to \infty$) the system is in its ground state with the smallest energy $E = -NFa$.

(b) From the relation between the total energy of the system and the extension of the chain, $E = -FL$ we have $\langle L \rangle = -\langle E \rangle / F$. Hence

$$\langle \ell \rangle = \frac{\langle L \rangle}{N} = \frac{ae^{\beta Fa} + be^{\beta Fb}}{e^{\beta Fa} + e^{\beta Fb}}. \tag{6}$$

Since $b < a$, in the large force regime (and with T not very high) one has

$$\begin{aligned} \langle \ell \rangle &= \frac{ae^{\beta Fa} + be^{\beta Fb}}{e^{\beta Fa}(1 + e^{\beta F(b-a)})} \\ &= \frac{a + be^{(b-a)\beta F}}{1 + e^{\beta F(b-a)}} \\ &\approx \left(a + be^{(b-a)\beta F}\right)\left(1 - e^{\beta F(b-a)}\right) \\ &= a + be^{(b-a)\beta F} - ae^{(b-a)\beta F} - be^{2(b-a)\beta F} \\ &\approx a - (a-b)e^{-(a-b)\beta F}, \end{aligned} \tag{7}$$

i.e. the average of the length per molecule is exponentially close to its maximum.

(c) In weak force regime ($\beta F a << 1$) we can develop the exponential terms in eq.6 to obtain

$$\begin{aligned}\langle \ell \rangle &\approx \frac{a+b+\beta F(a^2+b^2)}{2+\beta F(a+b)} \\ &= \frac{a+b}{2}\left(1+\beta F\left[\frac{a^2+b^2}{a+b}-\frac{a+b}{2}\right]\right) \\ &= \ell_0\left(1+\beta F\frac{(a-b)^2}{4\ell_0}\right) \qquad (8)\end{aligned}$$

By multiplying both sides of Eq.8 and calling $L_0 \equiv N\ell_0$ we can write

$$\langle L \rangle = L_0\left(1+\beta F N\frac{(a-b)^2}{4L_0}\right). \qquad (9)$$

Hence

$$\chi(T)(\langle L \rangle - L_0) = F \qquad (10)$$

with

$$\chi(T) = 4\frac{k_B T}{N(a-b)^2}. \qquad (11)$$

For a given force the average extension decreases with the temperature.

(d) The free energy per molecule of the system is given by

$$f(T,F) = -\frac{k_B T}{N}\ln Z_N(F,T) = -k_B T \ln \zeta(F,T) \qquad (12)$$

For $\beta F a << 1$ we have

$$\zeta = e^{\beta F a} + e^{\beta F b} \approx 2 + \beta F(a+b) + \frac{\beta^2 F^2}{2}(a^2+b^2) \qquad (13)$$

Hence

$$\ln \zeta \approx \ln 2 + \ln\left(1+\beta\frac{F}{2}(a+b)+\beta^2\frac{F^2}{4}(a^2+b^2)\right) \qquad (14)$$

ans since $\ln(1+t) \approx t - \frac{1}{2}t^2$ for $t << 1$, we have

$$\ln \zeta \approx \ln 2 + \beta F \ell_0 + \beta^2\frac{F^2}{8}(a-b)^2 \qquad (15)$$

In the regime of weak forces the free energy is

$$f(T,F) = -k_B T \ln \zeta \approx -k_B T \ln 2 - F\ell_0 - \beta\frac{F^2}{8}(a-b)^2 \qquad (16)$$

Note that, from Eq.8 we can write

$$\beta\frac{F}{4}(a-b)^2 = \langle \ell \rangle - \ell_0 \qquad (17)$$

Hence

$$f(T,F) \approx -k_B T \ln 2 - \frac{F}{2}\left(\langle \ell \rangle + \ell_0\right) \qquad (18)$$

A puzzling interacting theory

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Consider a theory for a real massless scalar field, characterized by the free Lagrangian, $\mathcal{L}_0 = \frac{1}{2}\left(\partial\phi\right)^2$, and by the following interaction term,

$$\mathcal{L}_{\rm int} = 2\lambda\phi\left(\partial\phi\right)^2 + 2\lambda^2\phi^2\left(\partial\phi\right)^2 \ .$$

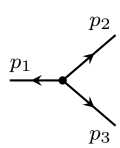
1. Compute the Feynman rules for the 3-point and 4-point interactions. *[7 points.]*
2. Compute the on-shell connected amplitude for the $\phi\phi \to \phi\phi$ scattering. *[8 points.]*
3. You should have found a peculiar result. Motivate it. *[5 points.]*

SOLUTIONS

There is not prefential way of deriving the Feynman rules. I'll do that using LSZ. I will also work in the convention where all 4-momenta are outgoing (to change to an incoming particle one just switches the sign of the corresponding 4-momentum). I'm also working in the "mostly minus" signature for the metric tensor. The S-matrix element for the 3-point interaction is,

$$\begin{aligned}\langle p_1p_2p_3|S|0\rangle &= \frac{1}{G(p_1)G(p_2)G(p_3)}\int d^4x_1d^4x_2d^4x_3\, e^{ip_1\cdot x_1+ip_2\cdot x_2+ip_3\cdot x_3}\,\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)e^{iS_{\rm int}}|0\rangle \\ &= \frac{4i\lambda}{G(p_1)G(p_2)G(p_3)}\int d^4x_1d^4x_2d^4x_3d^4y\, e^{ip_1\cdot x_1+ip_2\cdot x_2+ip_3\cdot x_3} \\ &\quad\times\Big[G(x_1-y)\,\partial G(x_2-y)\cdot\partial G(x_3-y)+G(x_2-y)\,\partial G(x_1-y)\cdot\partial G(x_3-y) \\ &\qquad+G(x_3-y)\,\partial G(x_1-y)\cdot\partial G(x_2-y)\Big]+\mathcal{O}(\lambda^2) \\ &= -4i\lambda\,(p_1\cdot p_2+p_1\cdot p_3+p_2\cdot p_3)\,(2\pi)^4\delta(p_1+p_2+p_3)\,,\end{aligned} \tag{1}$$

where in the second equality we have (a) expanded the interaction at $\mathcal{O}(\lambda)$, (b) used Wick's theorem to rewrite the vacuum expectation value in terms of the propagator, $G(x)$, and (c) neglected disconnected terms. In the third equality, instead, we integrated by parts first, and then used the definition of Fourier transformed propagator, $G(p)=\int d^4x\, e^{ip\cdot x}G(x)$. From the equation above we read the matrix element, $i\mathcal{M}$, i.e., the Feynman rule for the 3-point vertex,



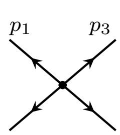
$$= -4i\lambda\,(p_1\cdot p_2+p_1\cdot p_3+p_2\cdot p_3)\ . \tag{2}$$

[Getting here is worth 3 points.]

One can proceed analogously to get the 4-point interaction vertex,

$$\begin{aligned}\langle p_1p_2p_3p_4|S|0\rangle &= \frac{1}{G(p_1)G(p_2)G(p_3)G(p_4)}\int d^4x_1d^4x_2d^4x_3d^4x_4\, e^{ip_1\cdot x_1+ip_2\cdot x_2+ip_3\cdot x_3+ip_4\cdot x_4} \\ &\quad\times\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{iS_{\rm int}}|0\rangle \\ &= \frac{8i\lambda^2}{G(p_1)G(p_2)G(p_3)G(p_4)}\int d^4x_1d^4x_2d^4x_3d^4x_4d^4y\, e^{ip_1\cdot x_1+ip_2\cdot x_2+ip_3\cdot x_3+ip_4\cdot x_4} \\ &\quad\times\Big[G(x_1-y)G(x_2-y)\partial G(x_3-y)\cdot\partial G(x_4-y)+G(x_1-y)G(x_3-y)\partial G(x_2-y)\cdot\partial G(x_4-y) \\ &\quad+G(x_1-y)G(x_4-y)\partial G(x_2-y)\cdot\partial G(x_3-y)+G(x_2-y)G(x_3-y)\partial G(x_1-y)\cdot\partial G(x_4-y) \\ &\quad+G(x_2-y)G(x_4-y)\partial G(x_1-y)\cdot\partial G(x_3-y)+G(x_3-y)G(x_4-y)\partial G(x_1-y)\cdot\partial G(x_2-y)\Big] \\ &= -8i\lambda^2\,(p_1\cdot p_2+p_1\cdot p_3+p_1\cdot p_4+p_2\cdot p_3+p_2\cdot p_4+p_3\cdot p_4)\,(2\pi)^4\delta(p_1+p_2+p_3+p_4)\,.\end{aligned} \tag{3}$$

The Feynman rule is then,

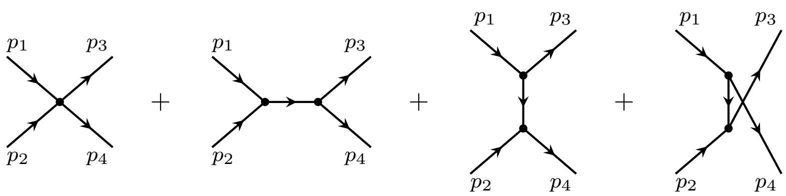


$$= -8i\lambda^2\,(p_1\cdot p_2+p_1\cdot p_3+p_1\cdot p_4+p_2\cdot p_3+p_2\cdot p_4+p_3\cdot p_4)\ . \tag{4}$$

[Getting here is worth 4 points.]

The connected amplitude for the $\phi\phi \to \phi\phi$ scattering is given by the following four diagrams,

$i\mathcal{M} =$



$$\equiv i\mathcal{M}_4 + i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_u \,. \tag{5}$$

Conservation of 4-momentum implies $p_1 + p_2 = p_3 + p_4$, while on-shellness implies $p_i^2 = 0$. These two things combined, imply,

$$p_1 \cdot p_2 = p_3 \cdot p_4 \,, \qquad p_1 \cdot p_3 = p_2 \cdot p_4 \,, \qquad p_1 \cdot p_4 = p_2 \cdot p_3 \,. \tag{6}$$

Moreover, the Mandelstam variables are,

$$s = (p_1 + p_2)^2 = p_1 \cdot p_2 \,, \qquad t = (p_1 - p_3)^2 = -p_1 \cdot p_3 \,, \qquad u = (p_1 - p_4)^2 = -p_1 \cdot p_4 \,. \tag{7}$$

The first diagram is obtained from Eq. (4) by flipping the sign of the incoming particles, i.e.,

$$i\mathcal{M}_4 = -8i\lambda^2 \left(p_1 \cdot p_2 - p_1 \cdot p_3 - p_1 \cdot p_4 - p_2 \cdot p_3 - p_2 \cdot p_4 + p_3 \cdot p_4\right) = -16i\lambda^2 (s + t + u) = 0 \,, \tag{8}$$

where we used the fact that $s + t + u = 4m^2 = 0$, since our bosons are massless.

[Getting here is 4 points.]

The other diagrams are obtained combining the 3-point vertex with the propagators,

$$\begin{aligned} i\mathcal{M}_s = &-32\lambda^2 \frac{\left[p_1 \cdot p_2 - p_1 \cdot (p_1 + p_2) - p_2 \cdot (p_1 + p_2)\right]\left[-(p_3 + p_4) \cdot p_3 - (p_3 + p_4) \cdot p_4 + p_3 \cdot p_4\right]}{(p_1 + p_2)^2} \\ = &-32\lambda^2 \frac{(p_1 \cdot p_2)(p_3 \cdot p_4)}{(p_1 + p_2)^2} = -32\lambda^2 s \,, \end{aligned} \tag{9a}$$

$$\begin{aligned} i\mathcal{M}_t = &-32\lambda^2 \frac{\left[-p_1 \cdot (p_1 - p_3) - p_1 \cdot p_3 + (p_1 - p_3) \cdot p_3\right]\left[p_2 \cdot (p_4 - p_2) - p_2 \cdot p_4 - (p_4 - p_2) \cdot p_4\right]}{(p_1 - p_3)^2} \\ = &-32\lambda^2 \frac{(p_1 \cdot p_3)(p_2 \cdot p_4)}{(p_1 - p_3)^2} = -32\lambda^2 t \,, \end{aligned} \tag{9b}$$

$$\begin{aligned} i\mathcal{M}_u = &-32\lambda^2 \frac{\left[-p_1 \cdot (p_1 - p_4) - p_1 \cdot p_4 + (p_1 - p_4) \cdot p_4\right]\left[p_2 \cdot (p_3 - p_2) - p_2 \cdot p_3 - (p_3 - p_2) \cdot p_3\right]}{(p_1 - p_4)^2} \\ = &-32\lambda^2 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3)}{(p_1 - p_4)^2} = -32\lambda^2 u \,, \end{aligned} \tag{9c}$$

so that, again, $i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_u = 0$. The on-shell connected amplitude at this order is thus simply zero.

[Getting here is worth 4 points.]

Indeed, one can show that the on-shell connected amplitude for the $\phi\phi \to \phi\phi$ scattering vanishes *at all orders*. This is because our theory is secretly nothing but a free theory written with an obscure field definition. Although one could just guess the right field redefintion, let us derive it systematically. We would like to find a field, $\psi \equiv \psi(\phi)$, such that, when written in terms of this new field, the total Lagrangian is simply $\mathcal{L}_0 + \mathcal{L}_{\rm int} = \frac{1}{2}(\partial\psi)^2$. This means to impose the following differential equation,

$$(\partial\psi)^2 = (\partial\phi)^2 + 4\lambda\phi(\partial\phi)^2 + 4\lambda^2\phi^2(\partial\phi)^2 \quad \Rightarrow \quad d\psi = \sqrt{1 + 4\lambda\phi + 4\lambda^2\phi^2}\, d\phi = (1 + 2\lambda\phi)\, d\phi \quad \Rightarrow \quad \psi = \phi + \lambda\phi^2 \,. \tag{10}$$

Given that a solution to the above differential equation exists (we just found it), and since observables are independent on field redefintions, any connected amplitude will be exactly zero.

[Getting here is worth 5 points.]