Problems

1 Classical Mechanics - Physics Basics of Curling

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In the sport of curling a heavy stone is thrown on an ice field, and the combination of translational and rotational motion of the stone determines its trajectory on the ice. Let us simplify the matter. We model the stone as a homogeneous ring (zero width) with radius R and total mass m.

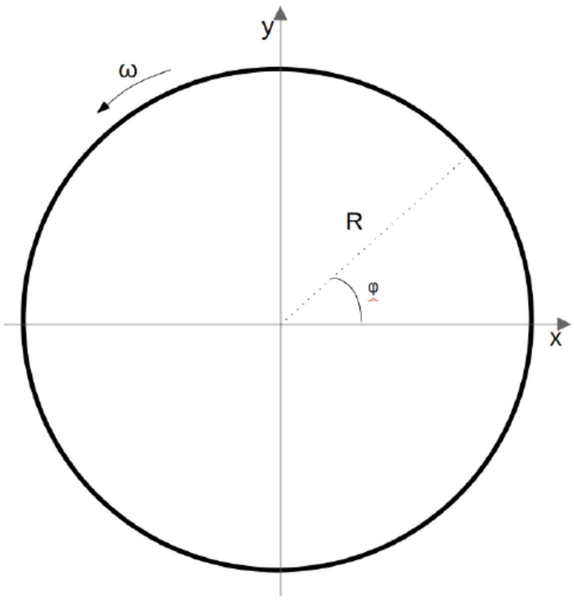

Figure 1: Curling stone as a homogeneous ring.

The stone is placed on the ice with its center of mass coinciding with the origin of a Cartesian axes system (x, y), and its angular velocity is ω (counter-clockwise). The center-of-mass velocity is initially zero. Figure 1 shows the situation.
The dynamic friction coefficient between the ice and the stone depends on the y-distance from the center of the stone as

$$\mu(y) = \mu_0 + \beta\,(y - y_{cm}),$$

where β is a positive constant.

Determine, at $t = 0$:

1. the friction force $\vec{F}_A$ acting on the stone; [6 points]
2. the torque $\vec{M}_A$ (moment of the force) generated by the friction force; [4 points]
3. the angular deceleration $\vec{\alpha}$. [4 points]

   Now imagine that the stone is in the same initial conditions as before (position and angular velocity), but with an initial velocity along the y-axis $\vec{v}_{cm}$.

4. In this case, calculate how the new friction force $\vec{F}'_A$ at $t = 0$ is different from $\vec{F}_A$, under the hypothesis that

$$|\vec{v}_{cm}| >> \omega R.$$

[6 points]

Congratulations! You now know (approximately) how to curl the stone and reach the goal. Now: exercise your sweeping and you'll be ready for the Olympics!

Solution

$$\vec{v} = \vec{\omega} \times \vec{R}$$

We de-compose $\vec{v}$ in its x and y coordinates:

$$v_x = -v\ sin\varphi = -\omega R\ sin\varphi$$
$$v_y = v\ cos\varphi = \omega R\ cos\varphi$$

We can now define the friction force on the infinitesimal mass dm:

$$d\vec{F}_A = -\ \mu\ g\ dm\ \hat{u}_v$$

where $\hat{u}_v = \frac{\vec{v}}{|\vec{v}|}$.
Switching to the component-like calculation:

$$dF_{A,x} = -\mu\ g\ dm\ \frac{v_x}{|v|} = \mu\ g\ sin\varphi\ dm$$
$$dF_{A,y} = -\mu\ g\ dm\ \frac{v_y}{|v|} = -\mu\ g\ cos\varphi\ dm$$

Considering that the infinitesimal mass of an omogeneous ring can be written as:

$$dm = \lambda\ R\ d\varphi = \frac{m}{2\pi R}\ R\ d\varphi = \frac{m}{2\pi}\ d\varphi$$

we can now substitute in the dF_A expressions:

$$dF_{A,x} = \frac{\mu g m}{2\pi}\ sin\varphi\ d\varphi$$
$$dF_{A,y} = -\frac{\mu g m}{2\pi}\ cos\varphi\ d\varphi\ .$$

We can now use the dependence of the friction coefficient with the y-distance to the stone's center (in this case equal to the axis system's origin, so $\mu(y) = \mu_0 + \beta y$):

$$dF_{A,x} = \frac{gm}{2\pi}\ (\mu_0 + \beta y)\ sin\varphi\ d\varphi = \frac{gm}{2\pi}\ (\mu_0 + \beta R\ sin\varphi)\ sin\varphi\ d\varphi$$
$$dF_{A,y} = -\frac{gm}{2\pi}\ (\mu_0 + \beta y)\ cos\varphi\ d\varphi = -\frac{gm}{2\pi}\ (\mu_0 + \beta R\ sin\varphi)\ cos\varphi\ d\varphi\ .$$

Integrating on φ:

$$F_{A,x} = \frac{\mu_0 g m}{2\pi} \int_0^{2\pi} sin\varphi\ d\varphi\ +\ \frac{gm\beta}{2\pi} R \int_0^{2\pi} sin^2\varphi\ d\varphi = \frac{1}{2} gm\beta R$$
$$F_{A,y} = \frac{\mu_0 g m}{2\pi} \int_0^{2\pi} cos\varphi\ d\varphi\ -\ \frac{gm\beta}{2\pi} R \int_0^{2\pi} sin\varphi\ cos\varphi\ d\varphi = 0\ .$$

$$\vec{F}_A = \frac{1}{2} gm\beta R\ \hat{u}_x + 0\ \hat{u}_y$$

We can now calculate the torque, starting from the infinitesimal and then integrating:

$$dM_A = -R\ dF_A = -R\ \mu\ g\ dm = -R\ \mu\ g \frac{m}{2\pi}\ d\varphi = -\frac{Rgm}{2\pi}\ (\mu_0 + \beta\ R\ sin\varphi)\ d\varphi$$

$$M_A = -\frac{Rgm}{2\pi}\ \left\{ \mu_0 \int_0^{2\pi} d\varphi\ +\ \beta R \int_0^{2\pi} sin\varphi\ d\varphi \right\} = -\mu_0 g m R$$

$$\vec{M}_A = -\mu_0 g m R\ \hat{u}_z$$

This means there is a constant angular deceleration equal to:

$$\alpha = \frac{M_A}{I} = -\frac{\mu_0 gmR}{mR^2} = -\frac{\mu_0 g}{R}$$

Considering now the initial velocity along the y axis $\vec{v}_{cm}$:

$$v'_x = -\omega R\ sin\varphi$$
$$v'_y = v_{cm} + \omega R\ cos\varphi$$

Now, calculating v' and applying the required approximation:

$$v' = \sqrt{v'^2_x + v'^2_y} = \sqrt{\omega^2 R^2\ sin^2\varphi + v^2_{cm} + \omega^2 R^2\ cos^2\varphi + 2v_{cm}\omega R\ cos\varphi} \simeq \sqrt{v_{cm}(v_{cm} + 2\omega R\ cos\varphi)} \simeq v_{cm}$$

Therefore, in this approximationç

$$\frac{v'_x}{v'} \simeq -\frac{\omega\ R\ sin\varphi}{v_{cm}} \qquad \frac{v'_x}{v'} \simeq 1$$

Which brings us to:

$$dF'_{A,x} = -\mu\ g\ dm\ \left(-\frac{\omega R\ sin\varphi}{v_{cm}}\right) = \mu\ g\ \frac{m}{2\pi R}\ R\ d\varphi\ \frac{\omega R\ sin\varphi}{v_{cm}} = \frac{gm\omega R}{2\pi v_{cm}}(\mu_0 + \beta\ R\ sin\varphi)\ sin\varphi\ d\varphi$$
$$dF'_{A,y} = -\mu\ g\ dm \cdot 1 = -mug\frac{m}{2\pi R}R\ d\varphi = -\frac{gm}{2\pi}(\mu_0 + \beta\ R\ sin\varphi)\ d\varphi\ .$$

Integrating over φ:

$$F'_{A,x} = \frac{gm\omega R}{2\pi v_{cm}} \cdot \left\{\mu_0 \int_0^{2\pi} sin\varphi\ d\varphi + \beta R \int_0^{2\pi} sin^2\varphi\ d\varphi\right\} = \frac{1}{2}gm\beta R\frac{\omega R}{v_{cm}} = F_{A,x} \cdot \frac{\omega R}{v_{cm}}$$
$$F'_{A,y} = -\frac{gm}{2\pi} \cdot \left\{\mu_0 \int_0^{2\pi} d\varphi + \beta R \int_0^{2\pi} sin\varphi\ d\varphi\right\} = -\frac{gm}{2\pi}\mu_0 2\pi = -\mu_0 gm\ .$$

As expected, the force along the y-axis corresponds to the usual friction force for the whole mass (the modulation of μ with y is linear, so the average μ is μ_0). On the x-axis the difference with respect to the case with no motion on y is that the “small” $\frac{\omega R}{v_{cm}}$ factor appears.

2 Classical Mechanics - Earth is (Almost) Round

Prof. Claudio Dappiaggi - University of Pavia

The surface of Earth can be approximated with very good precision by a 2-sphere S^2. At page 10 of volume 1 of the (British) Admiralty Manual of Navigation, it is written that:

> "The errors introduced by assuming a spherical Earth based on the international nautical mile are not more than 0.5% for latitude, 0.2% for longitude."

This exercise aims at validating this statement.
Hence, considering a sphere S^2 of radius $R > 0$:

1. Prove that the shortest path connecting two points thereon is a *great circle* (or *orthodrome*), namely the circular intersection between S^2 and a plane passing through its center point. [10 points]

2. Construct the *haversine formula* that computes the distance d between two points on the sphere as a function of their latitude and longitude. [8 points]

3. Setting the radius of Earth as 6378 km and assuming the distance between the cities of Oporto and New York to be 5465 km [1], estimate the error using the haversine formula, knowing that the coordinates of Oporto are 41.147658° latitude, −8.674770° longitude, while those of New York are 40.712776° latitude and −74.005974° longitude. [2 points]

Hint: Recall that, in absence of external forces, a point particle travelling between two points always follows the shortest path.

[1] Source: https://www.searates.com/distance-time/

Pre-Plancks 2025

Exercise 1

Earth is (almost) round

The surface of Earth can be approximated with such a good precision by a 2-sphere $\mathbb{S}^2$ that, at page 10 of volume 1 of the (British) Admiralty Manual of Navigation, it is written that *"The errors introduced by assuming a spherical Earth based on the international nautical mile are not more than 0.5% for latitude, 0.2% for longitude".* This exercise aims at validating this statement. Hence, considering a sphere $\mathbb{S}^2$ of radius $R > 0$

- prove that the shortest path connecting two points thereon is a *great circle (or orthodrome)*, namely the circular intersection between $\mathbb{S}^2$ and a plane passing through its center point [**5 Points**],
- Construct the *haversine formula* which computes the distance d between two points on the sphere as a function of their latitude and longitude [**4 Points**]
- Setting the radius of Earth as 6378Km and assuming the distance between the cities of Oporto and New York to be 5465Km (source), estimate the error using the haversine formula knowing that the coordinates of Oporto are 41.147658° latitude, -8.674770° longitude., while those of New York are 40.712776° latitude and -74.005974° longitude. [**1 Point**]

[Hint: Recall that, in absence of external forces, a point particle traveling between two points always follows the shortest path.]

Solution: *1.* – Consider a sphere of radius $R > 0$ realized in $\mathbb{R}^3$, namely $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = R^2\}$. Introducing the spherical coordinates. $R > 0$, $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$,

$$\begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases}, \tag{1}$$

we can use the hint to infer that the shortest path connecting two points on the sphere is the trajectory followed by a point particle thereon in absence of external forces. Assuming for simplicity that it has mass $m = 1$, although the value of m is irrelevant, we can infer that the Lagrangian is constituted only by the kinetic term, namely

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{mR^2}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2),$$

where we used that

$$\begin{cases} \dot{x} = R[\cos\theta\cos\varphi\dot{\theta} - \sin\theta\sin\varphi\dot{\varphi}] \\ \dot{y} = R[\cos\theta\sin\varphi\dot{\theta} + \sin\theta\cos\varphi\dot{\varphi}] \\ \dot{z} = -R\sin\theta\dot{\theta} \end{cases} .$$

Observing that the coordinate φ is cyclic and that, being any time-dependent constraint absent, the energy is conserved, we can infer that there exists two constants E, M depending on the initial data such that

$$\begin{cases} \dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2 = E \\ \sin^2\theta\dot{\varphi} = L \end{cases} \implies \begin{cases} \dot{\theta} = \pm\sqrt{2E - \frac{L}{\sin^2\theta}} \\ \dot{\varphi} = \frac{L}{\sin^2\theta} \end{cases}$$

Choosing arbitrary one of the signs of the square root since the proof does not depend on it, we observe that the term under square root is always non-negative. That said, we can consider the phase portrait of this system of ODEs, namely

$$\frac{d\varphi}{d\theta} = \frac{M}{\sin\theta\sqrt{2E\sin^2\theta - L^2}}.$$

Considering a generic initial point (θ_0, φ_0), the solution of this equation reads

$$\varphi - \varphi_0 = \int_{\theta_0}^{\theta} dx \, \frac{M}{\sin(x)\sqrt{2E\sin^2 x - M^2}} = \int_{\theta_0}^{\theta} \frac{dx}{\sin^2 x} \, \frac{M}{\sqrt{2E - \frac{M^2}{\sin^2 x}}}.$$

By observing that $\frac{dx}{\sin^2 x} = -d(\cot(x))$, we change variables as $y = \cot(x)$. Hence, recalling that $\sin^{-2} x = 1 + \cot^2 x$,

$$\varphi - \varphi_0 = \int_{\cot\theta}^{\cot\theta_0} dy \, \frac{M}{\sqrt{2E - M^2 - M^2 y^2}} = \int_{\cot\theta}^{\cot\theta_0} dy \, \frac{1}{\sqrt{K^2 - y^2}},$$

where $K^2 = \frac{2E - M^2}{M^2} > 0$. At last, setting $\tilde{y} \doteq \frac{y}{K}$, we get to the final expression

$$\varphi - \varphi_0 = \int_{\frac{\cot\theta}{K}}^{\frac{\cot\theta_0}{K}} d\tilde{y} \frac{1}{\sqrt{1 - \tilde{y}^2}} = \arcsin\tilde{y}\Big|_{\frac{\cot\theta}{K}}^{\frac{\cot\theta_0}{K}} .$$

This can be rewritten as

$$\frac{\cot\theta}{K} = \sin(\varphi - \tilde{\varphi}_0) = \sin\varphi\cos\tilde{\varphi}_0 - \cos\varphi\sin\tilde{\varphi}_0,$$

where $\tilde{\varphi}_0 = \varphi_0 + \arcsin(\frac{\cot\theta_0}{K})$. Switching back to Cartesian coordinates this equation becomes

$$\frac{z}{\sqrt{x^2 + y^2}} = K\left(\frac{y}{\sqrt{x^2 + y^2}}\cos\tilde{\varphi}_0 - \frac{x}{\sqrt{x^2 + y^2}}\sin\tilde{\varphi}_0\right),$$

namely

$$z = K(y\cos\tilde{\varphi}_0 - x\sin\tilde{\varphi}_0),$$

which is the equation of a plane passing through the center point of the sphere.

2. Having established in the first point that the distance between two points on a sphere is an arc of a circumference, denoting p, q the two points, it turns out that $d_{pq} = R\alpha$, where α is the angle which the arc connecting p to q subtends at the center of the circle. Since we want to express α as a function of longitude and latitude, hence of the spherical coordinates, we observe that, calling $d_E(p,q)$ the Euclidean distance between p and q, Carnot theorem yields

$$2R\sin\frac{\alpha}{2} = d_E(p,q) = [(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2]^{\frac{1}{2}}. \tag{2}$$

Using Equation (1) and replacing the subscript p with 1 and q with 2, it turns out that

$$d_E^2(p,q) = R^2\left[(\cos\theta_2\cos\varphi_2 - \cos\theta_1\cos\varphi_1)^2 + (\cos\theta_2\sin\varphi_2 - \cos\theta_1\sin\varphi_2)^2 + \right.$$
$$\left. + (\sin\theta_2 - \sin\theta_1)^2\right] = 2R^2\left[1 - \cos\theta_1\cos\theta_2\cos(\varphi_2 - \varphi_1) - \sin\theta_1\sin\theta_2\right] \tag{3}$$

Recalling that

$$\sin\alpha = \sin\left(2\frac{\alpha}{2}\right) = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\sin\frac{\alpha}{2}\sqrt{1-\sin^2\frac{\alpha}{2}},$$

Equation (2) entails that

$$\alpha = \arcsin\left(\frac{d}{R}\sqrt{1-\frac{d^2}{4R^2}}\right).$$

Replacing Equation (3) in this expression, we end up with

$$\alpha = 2\arcsin\left(\sqrt{\sin^2\frac{\theta_2-\theta_1}{2} + \cos\theta_2\cos\theta_1\sin^2\frac{\varphi_2-\varphi_1}{2}}\right),$$

from which it descends the *haversine distance formula*[1]:

$$d_{pq} = 2R\arcsin\sqrt{\frac{1-\cos(\theta_2-\theta_1)+\cos\theta_2\cos\theta_1(1-\cos(\varphi_2-\varphi_1))}{2}}. \tag{4}$$

3. In order to use Equation (4), we need to recall that θ corresponds to the latitude and φ to the longitude. Transforming the data in radians yields (approx.)

$$\theta_1 \simeq 0,2286\pi, \quad \theta_2 \simeq 0,2262\pi, \quad \varphi_1 \simeq -0,0482\pi, \quad \varphi_2 \simeq -0,4111\pi.$$

Replacing these data in Equation (4) with the radius of Earth yields

$$\text{Distance Oporto to New York} \simeq 5357Km,$$

which entails an error around 1.8% with respect to the given distance. Observe that we <u>cannot</u> use Google to compute the real distance since it uses Equation (4) and you get exactly 5357Km.

[1]The name haversine comes from the fact that the expression can be rewritten in terms of the haversine function $\mathrm{hav}(\theta) = \sin^2\frac{\theta}{2}$.

3 Electromagnetism - The line and the cylinder

Prof. Giancarlo Maero - University of Milan

An infinite, straight line of charge with uniform linear charge density λ is placed at a distance $D > R_w$ from the axis of an infinitely long, perfectly conducting cylinder of radius R_w. The conductor is neutral and floating. We want to find the electrostatic potential in the domain represented by the whole space $\mathbb{R}^3$ minus the cylinder, using the method of image charges.

1. To do so, first choose a proper image charge distribution to solve the equivalent problem and reason about a suitable option for the magnitude of the image and the integration constants in the expression of the potential. [4 points]
2. Determine the position of the image charge distribution, coherently with the boundary conditions. [4.5 points]
3. Using the results obtained so far, rewrite the definitive expression of the potential, first at any point in the domain and then on the cylinder surface. Given $\lambda = 1\,\mathrm{nC/m}$, $R_w = 45\,\mathrm{mm}$ and $D = 2R_w$, calculate the numerical value of the potential on the cylinder. [2.5 points]
4. Determine the expression of the charge distribution induced by λ on the cylinder. Sketch a qualitative diagram of the distribution versus the position on the cylinder, and determine the position and value (both expression and numbers, using the data given in 3.) of its maxima and minima (absolute value). How do you expect the interaction force to be between the line charge and the cylinder? [4 points]
5. Based on the result obtained above, and without further calculations, deduce what happens if the line of charge λ is inside a cylindrical empty shell of conductor, i.e. $D < R_w$ (hence the problem's domain is $r < R_w$). What is the value and position of the image? [2 points]
6. Consider the latter case, i.e. $D < R_w$: A line of charge is placed inside a cylindrical conductive shell in vacuum. Add a uniform magnetic induction field $\vec{B} = B\hat{e}_z$, with $\hat{e}_z$ the cylinder's axis. It can be shown that the line of charge undergoes a rigid motion (as if it were a single body) with velocity $\vec{v} = \vec{E} \times \vec{B}/B^2$. Show that $\vec{v}$ has the form $\vec{v}(D) = v(D)\,\hat{e}_\theta$. Determine the full expression of $\vec{v}$ and of the angular rotation frequency ω. Considering $\lambda = 1\,\mathrm{nC/m}$, $B = 0.15\,\mathrm{T}$, $R_w = 45\,\mathrm{mm}$ and $D = R_w/2$, calculate the value of ω. [3 points]

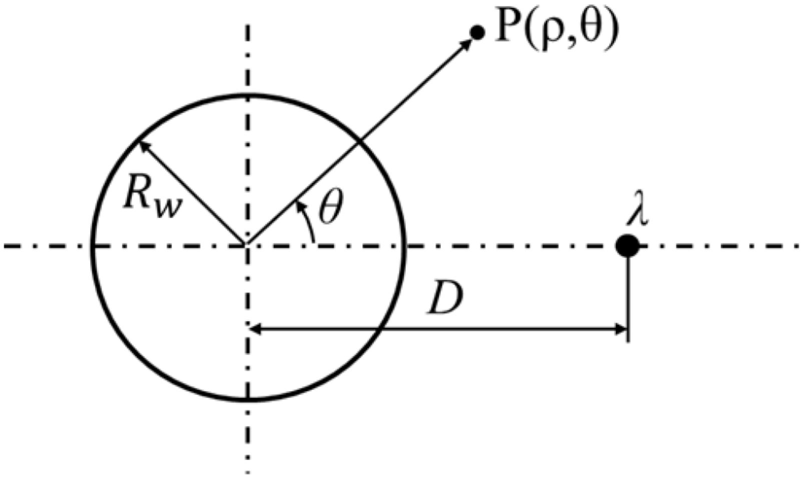

Figure 2: Sketch of the problem. The line of charge λ is placed in the vicinity of a cylindrical conductor.

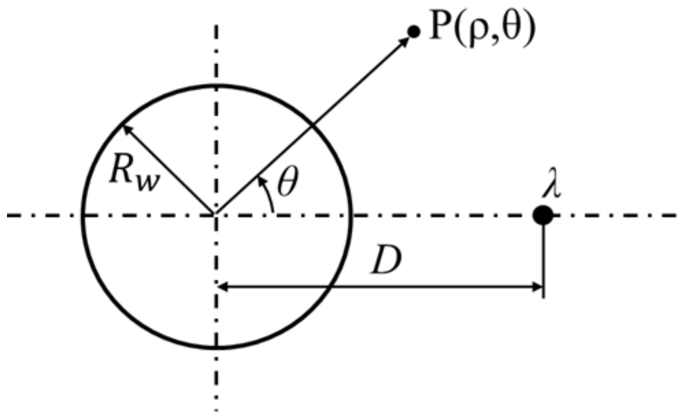

Figure 1: Sketch of the problem. The line of charge λ is placed in the vicinity of a cylindrical conductor.

Solution

1. Referring to Fig. 2, we can say that the electric field for $\lambda\,(D,0)$ in a generic point $P\,(\rho,\theta)$ would be

$$\overline{E}_{real}(P) = \frac{\lambda}{2\pi\varepsilon_o r_1}\widehat{r}_1, \tag{1}$$

where $\overline{r}_1$ the vector from the source to P, whose potential is

$$\Phi_{real}(P) = -\frac{\lambda}{2\pi\varepsilon_o}\log(r_1) + C_1, \tag{2}$$

A suitable image is another line charge $\lambda'\,(D',0)$, with $D' < R_w$, whose field and potential read

$$\overline{E}_{img}(P) = \frac{\lambda'}{2\pi\varepsilon_o r_2}\widehat{r}_2, \quad \Phi_{img}(P) = -\frac{\lambda'}{2\pi\varepsilon_o}\log(r_2) + C_2, \tag{3}$$

where again where $\overline{r}_2$ is the vector from the image to P. Constants may be determined by reasoning about proper reference points: We notice that the overall potential reads

$$\Phi(P) = \Phi_{real}(P) + \Phi_{img}(P) = -\frac{\lambda}{2\pi\varepsilon_o}\log(r_1) - \frac{\lambda'}{2\pi\varepsilon_o}\log(r_2) + C_1 + C_2, \tag{4}$$

and a reasonable combination of λ', C_1, C_2 allows us to obtain a zero potential on the midplane between λ and λ', that is

$$\lambda' = -\lambda, \quad C_1 + C_2 = 0, \tag{5}$$

so that

$$\Phi(P) = \frac{\lambda}{2\pi\varepsilon_0}\log\frac{r_2}{r_1}. \tag{6}$$

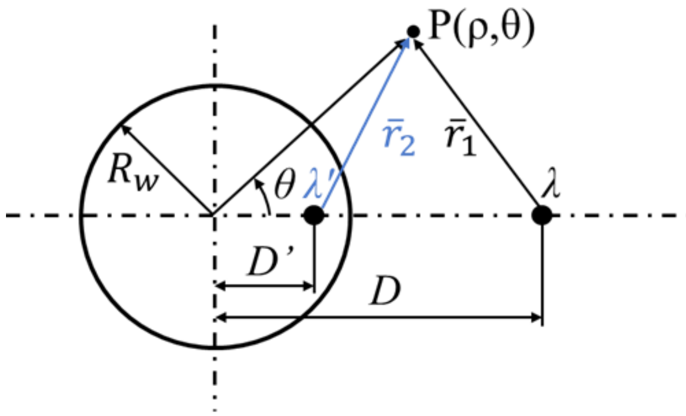

Figure 2: Sketch of the problem's solution, with real and image line charges.

2. To determine the position of λ', consider that the potential has the same value for any point on its surface, $\Phi\left(R_w, \theta\right) = constant$. We conveniently write, for any $P\left(\rho, \theta\right)$

$$\begin{aligned} r_1^2 &= D^2 + \rho^2 - 2D\rho\cos\theta \\ r_2^2 &= D'^2 + \rho^2 - 2D'\rho\cos\theta \end{aligned} \tag{7}$$

and argue that the argument r_2/r_1 of the logarithm in the expression for Φ should not depend on θ when $\rho = R_w$. With some algebra we rewrite this ratio as

$$\begin{aligned} \left(\frac{r_2^2}{r_1^2}\right)^{1/2} &= \left(\frac{D'^2 + R_w^2 - 2D'R_w\cos\theta}{D^2 + R_w^2 - 2DR_w\cos\theta}\right)^{1/2} = \\ = \left\{\frac{D'^2 + R_w^2}{D^2 + R_w^2}\left[1 - \frac{2D'R_w\cos\theta}{D'^2 + R_w^2}\right] / \left[1 - \frac{2DR_w\cos\theta}{D^2 + R_w^2}\right]\right\}^{1/2} &= \\ = \left\{\frac{D'^2 + R_w^2}{D^2 + R_w^2}\left[1 - f(\theta)\right] / \left[1 - g(\theta)\right]\right\}^{1/2} \end{aligned} \tag{8}$$

and for this quantity to be independent of θ we can enforce $f(\theta) = g(\theta)$, i.e.

$$\frac{2D'R_w\cos\theta}{D'^2 + R_w^2} = \frac{2DR_w\cos\theta}{D^2 + R_w^2}, \tag{9}$$

which, after some elementary algebra, reads

$$(R_w^2 - DD')(D - D') = 0. \tag{10}$$

The solution $D' = D$ is not meaningful, therefore we conclude that

$$D' = \frac{R_w^2}{D}. \tag{11}$$

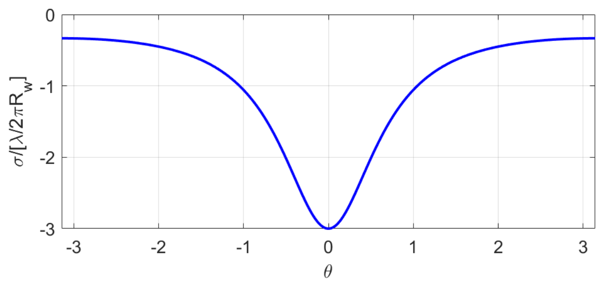

Figure 3: Diagram of the induced charge density σ vs θ on the cylinder surface.

3. We get

$$\begin{aligned}\Phi(\rho,\theta) &= \frac{\lambda}{2\pi\varepsilon_0}\log\frac{r_2}{r_1} = \frac{\lambda}{4\pi\varepsilon_0}\log\frac{r_2^2}{r_1^2} = \\ &= \frac{\lambda}{4\pi\varepsilon_0}\log\left[\frac{D'^2+\rho^2-2D'\rho\cos\theta}{D^2+\rho^2-2D\rho\cos\theta}\right] = \\ &= \frac{\lambda}{4\pi\varepsilon_0}\log\left[\frac{R_w^4/D^2+\rho^2-2\rho(R_w^2/D)\cos\theta}{D^2+R_w^2-2DR_w\cos\theta}\right]\end{aligned} \tag{12}$$

and with minimal algebra

$$\Phi(R_w,\theta) = \frac{\lambda}{2\pi\varepsilon_0}\log\frac{R_w}{D} = -12.46\ V. \tag{13}$$

4. The surface distribution of induced charge is proportional to the normal electric field on the surface itself,

$$\begin{aligned}\sigma(R_w,\theta) &= \varepsilon_0 E_\rho(R_w,\theta) = -\varepsilon_0\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=R_w} = \\ &= -\frac{\lambda}{4\pi}\frac{\partial}{\partial\rho}\left[\log\left(\rho^2+(R_w^4/D^2)-2\rho(R_w^2/D)\cos\theta\right)-\log\left(\rho^2+D^2-2\rho D\cos\theta\right)\right]\Big|_{\rho=R_w} = \\ &= -\frac{\lambda}{4\pi}\left[\frac{2\rho-2(R_w^2/D)\cos\theta}{\rho^2+(R_w^4/D^2)-2\rho(R_w^2/D)\cos\theta}-\frac{2\rho-2D\cos\theta}{\rho^2+D^2-2\rho D\cos\theta}\right]\Bigg|_{\rho=R_w} = \ldots\end{aligned} \tag{14}$$

and with agonizing algebra we get

$$\sigma(R_w,\theta) = -\frac{\lambda}{2\pi R_w}\frac{(D/R_w)^2-1}{1+(D/Rw)^2-2(D/R_w)\cos\theta} < 0\ \forall\theta. \tag{15}$$

whose diagram is sketched in Fig. 3. The induced charge is negative at all places, and we expect it to interact with the line of charge through an attractive force. Likewise, in the equivalent problem, we can see an attractive force exchanged between the real source and the opposite-signed image.

The extreme charge density values $|\sigma|_{max,min}$ are obtained for $\theta = 0, \pi$, respectively, with values

$$
\begin{aligned}
|\sigma|_{max} = |\sigma(\theta = 0)| &= \frac{\lambda}{2\pi R_w} \frac{d/Rw + 1}{d/Rw - 1} = 1.06 \cdot 10^{-8}\ C/m^2, \\
|\sigma|_{min} = |\sigma(\theta = \pi)| &= \frac{\lambda}{2\pi R_w} \frac{d/Rw - 1}{d/Rw + 1} = 1.18 \cdot 10^{-9}\ C/m^2
\end{aligned}
\tag{16}
$$

5. If the domain is the inside of the conductor (supposed to be a closed cylindrical shell of perfect conductor with negligible wall thickness) and the charge λ is inside the shell at a position $(D, 0)$ with $D < R_w$, nothing changes as following the reasoning in 1. we have an image $\lambda' = -\lambda$ outside and the roles of real and image charge, D and D' are simply swapped, with $D' = R_w^2/D$ now being $< R_w$.

6. The velocity of the line of charge is said to be $\overline{v} = \overline{E} \times \overline{B}/B^2$. The field $\overline{E}$ acting on λ is that of the induced charge, or equivalently, the one exerted by the image $-\lambda$ and thus the field of a line of charge. The field is directed along the line between image and source, i.e. the radial direction, at a distance $h = D' - D = (R_w^2/D - D) = (R_w^2 D^2)/D^2$. The angular position θ of λ has no impact, so the velocity is purely a function of the radial position. We conclude that the velocity vector takes the form

$$
\overline{v}(D) = \overline{E} \times \overline{B}/B^2 = \frac{1}{B} E_\rho(D) \widehat{e}_\rho \times \widehat{e}_z = -\frac{E_\rho(D)}{B} \widehat{e}_\theta \tag{17}
$$

hence the motion is a rotation around the cylinder's axis, with speed depending on the radial offset.

In explicit terms, the field reads, at the position $(D, 0)$ of the real charge,

$$
\overline{E}_{img}(D, 0) = \frac{-\lambda}{2\pi\varepsilon_o} \frac{1}{h} (-\widehat{e}_\rho) = \frac{\lambda}{2\pi\varepsilon_o R_w^2} \frac{D}{1 - (d/R_w)^2} \widehat{e}_\rho. \tag{18}
$$

Therefore we get a velocity vector

$$
\overline{v}(D) = -\frac{\lambda}{2\pi\varepsilon_o B R_w^2} \frac{D}{1 - (d/R_w)^2} \widehat{e}_\theta \tag{19}
$$

and the angular rotation frequency reads

$$
\omega(D) = \frac{v_\theta}{D} = -\frac{\lambda}{2\pi\varepsilon_o B R_w^2} \frac{1}{1 - (D/R_w)^2} = (-)1.26 \cdot 10^4\ rad/s. \tag{20}
$$

NOTE: There is at least another approach to solve Point 2. If on the cylinder surface we ask for a constant value V, then

$$\begin{aligned}\frac{\lambda}{4\pi\varepsilon_o}\log\left(\frac{D'^2+R_w^2-2D'R_w\cos\theta}{D^2+R_w^2-2DR_w\cos\theta}\right) &= V \Rightarrow \\ \log(...) = 4\pi\varepsilon_0 V/\lambda &\Rightarrow \\ (...) = \exp(4\pi\varepsilon_0 V/\lambda) = k^2 &\Rightarrow \\ D'^2-k^2D^2+R_w^2(1-k^2) = 2R_w(D'^2-k^2D^2)\cos\theta &\end{aligned} \tag{21}$$

and since at the right hand side there is an expression $2R_w(D'^2-k^2D^2)$ multiplied times the periodic function $\cos\theta$, both such expression and the whole left hand side must vanish separately in order to be equal $\forall\theta$. Setting them both equal to zero yields, with some algebra, $k = R_w/d$ and $D' = R_w^2/D$.

Appendix: Alternative Solution for a Neutral Floating Cylinder

This section provides an alternative derivation for the case of a **neutral and floating** conductor, as requested in the problem statement. To ensure the net charge on the cylinder is zero, we introduce an additional image charge $\lambda_c = +\lambda$ at the origin $(0, 0)$ to cancel the charge of the primary image $\lambda' = -\lambda$.

1. Image Charge Selection for Neutrality

To solve the equivalent problem for a neutral cylinder, we apply the method of image charges by superimposing three line charges:

- The real line charge λ at distance D from the axis.
- A primary image line charge $\lambda' = -\lambda$ at distance D' to make the surface an equipotential.
- A second image line charge $\lambda_c = +\lambda$ at the origin $(0, 0)$ to preserve the neutrality of the conductor ($Q_{net} = \lambda' + \lambda_c = 0$).

The total potential $\Phi(P)$ at a point $P(\rho, \theta)$ is the sum of the potentials from each source. Using R_w as the reference distance for all charges to maintain a consistent constant of integration, the potential reads:

$$\Phi(P) = -\frac{\lambda}{2\pi\epsilon_o}\ln\left(\frac{r_1}{R_w}\right) - \frac{\lambda'}{2\pi\epsilon_o}\ln\left(\frac{r_2}{R_w}\right) - \frac{\lambda_c}{2\pi\epsilon_o}\ln\left(\frac{\rho}{R_w}\right) \tag{1}$$

Substituting $\lambda' = -\lambda$ and $\lambda_c = \lambda$, we apply logarithmic identities:

$$\Phi(P) = \frac{\lambda}{2\pi\epsilon_o}\left[\ln\left(\frac{r_2}{R_w}\right) - \ln\left(\frac{r_1}{R_w}\right) - \ln\left(\frac{\rho}{R_w}\right)\right] = \frac{\lambda}{2\pi\epsilon_o}\ln\left(\frac{r_2}{r_1}\right) - \frac{\lambda}{2\pi\epsilon_o}\ln\left(\frac{\rho}{R_w}\right) \tag{2}$$

Expressing r_1 and r_2 in polar coordinates via the law of cosines ($r^2 = \rho^2 + d^2 - 2\rho d\cos\theta$), the full expression is:

$$\Phi(\rho, \theta) = \frac{\lambda}{4\pi\epsilon_o}\ln\left[\frac{D'^2 + \rho^2 - 2D'\rho\cos\theta}{D^2 + \rho^2 - 2D\rho\cos\theta}\cdot\frac{R_w^2}{\rho^2}\right] \tag{3}$$

2. Position of the Image Charges

For the cylinder surface $\rho = R_w$ to be an equipotential, the ratio r_2/r_1 must be independent of θ. As shown in the primary derivation, this condition is satisfied when:

$$D' = \frac{R_w^2}{D} \quad \text{and} \quad \frac{r_2}{r_1} = \frac{R_w}{D} \tag{4}$$

The term $-\frac{\lambda}{2\pi\epsilon_o}\ln(\rho/R_w)$ only depends on ρ and becomes zero at $\rho = R_w$. Thus, the central image does not affect the shape of the equipotential surface, and the boundary condition remains satisfied.

3. Expression of the Potential and Numerical Value

On the cylinder surface ($\rho = R_w$), the term $\ln(\rho/R_w)$ vanishes. The potential of the conductor is determined solely by the first two charges:

$$\Phi(R_w) = \frac{\lambda}{2\pi\epsilon_o}\ln\left(\frac{R_w}{D}\right) \tag{5}$$

Using the provided values ($\lambda = 1$ nC/m, $R_w = 45$ mm, and $D = 2R_w$):

$$\Phi(R_w) = \frac{10^{-9}}{2\pi\cdot 8.854\times 10^{-12}}\ln\left(\frac{1}{2}\right) \approx -12.46 \text{ V} \tag{6}$$

4. Induced Charge Distribution

The surface charge density $\sigma(\theta)$ is obtained from the radial gradient of the total potential at the conductor surface:

$$\sigma(R_w,\theta) = -\epsilon_0 \left.\frac{\partial\Phi_{total}}{\partial\rho}\right|_{\rho=R_w} = -\epsilon_0 \left[\frac{\partial\Phi_{ext}}{\partial\rho} + \frac{\partial\Phi_{cent}}{\partial\rho}\right]_{\rho=R_w} \tag{7}$$

The first term corresponds to the non-neutral distribution induced by the external line charge and its primary image. The second term, derived from the central image charge $\lambda_c = +\lambda$, adds a uniform positive density $\frac{\lambda}{2\pi R_w}$ to ensure the total net charge is zero.

Combining these contributions, we obtain:

$$\sigma(R_w,\theta) = \frac{\lambda}{2\pi R_w}\left[1 - \frac{(D/R_w)^2 - 1}{1 + (D/R_w)^2 - 2(D/R_w)\cos\theta}\right] \tag{8}$$

By finding a common denominator and simplifying the numerator, the expression can be rewritten in the more compact form:

$$\sigma(R_w,\theta) = \frac{\lambda}{\pi R_w}\frac{1 - \frac{D}{R_w}\cos\theta}{1 + (\frac{D}{R_w})^2 - 2\frac{D}{R_w}\cos\theta} \tag{9}$$

For the case $D = 2R_w$:

- **Maximum (absolute value) at $\theta = 0$:** $\sigma(0) = \frac{\lambda}{2\pi R_w}(1-3) = -7.07 \times 10^{-9}$ C/m^2.
- **Minimum (absolute value) at $\theta = \pi$:** $\sigma(\pi) = \frac{\lambda}{2\pi R_w}(1-1/3) = +2.36 \times 10^{-9}$ C/m^2.

The presence of positive induced charge on the far side ($\theta = \pi$) confirms that the total charge $\oint \sigma R_w d\theta = 0$, satisfying the neutrality condition of the floating conductor.

The force per unit length $\vec{f}$ on the real line charge λ is the sum of the attraction from λ' and the repulsion from λ_c:

$$\vec{f} = \frac{\lambda^2}{2\pi\epsilon_o}\left[\frac{1}{D} - \frac{D}{D^2 - R_w^2}\right]\hat{e}_\rho \tag{10}$$

Since $D/(D^2 - R_w^2) > 1/D$ for all $D > R_w$, the net force remains **attractive** toward the cylinder.

4 Statistical Mechanics - 1D crystal in equilibrium at temperature T

Prof. Amos Maritan - University of Padova

Consider a one-dimensional array of $N+1$ particles of mass m aligned along the x-axis. The Hamiltonian of the system is:

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + u\big((x_i - x_{i-1})\big) \right]$$

where x_i is the position of the i-th particle, p_i is its conjugate momentum and

$$u(z) = \alpha z^4 \text{ if } z > 0$$

whereas $u(z) = +\infty$ if $z < 0$. The first particle is held fixed at the origin $x_0 = 0$.
The system is in thermal equilibrium at temperature T in contact with a heat bath (canonical ensemble). α is a positive constant.

1. Can we consider the particle indistinguishable? [2 points]
2. Determine the Helmholtz free energy F (hint: use the definition of Gamma function $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z}\, dz$), [5 points]
3. the average energy $E = \langle H \rangle$, [3 points]
4. the entropy S [3 points]
5. the average length of the chain

$$L \equiv \left\langle \sum_{i=1}^{N} (x_i - x_{i-1}) \right\rangle$$

[3 points]

6. If in the above problem, we consider the potential

$$u(z) = \alpha z^4 \qquad \forall z \in \mathbb{R},$$

how would the average length of the chain be? Comment the answer calculating

$$L_2 = \sum_{i=1}^{N} \sqrt{\langle (x_i - x_{i-1})^2 \rangle}.$$

[6 points]

Solution

1. The particles occupy definite positions along the chain and are distinguishable because of their order along the chain and the fixed boundary condition at $x_0 = 0$. Swapping two particles changes the physical configuration.

2. In the canonical ensemble,

$$Z = \int \prod_{i=1}^{N} \frac{dp_i\, dx_i}{h}\, e^{-\beta H}, \quad \beta = 1/(\kappa_B T)$$

Since the Hamiltonian is a sum of a purely kinetic part $\sum_i p_i^2/(2m)$ and a purely configurational part $\sum_i u(x_i - x_{i-1})$, the partition function factorizes as

$$Z = Z_p\, Z_x / h^N,$$

with

$$Z_p = \int \prod_{i=1}^{N} dp_i\, e^{-\beta \sum_i p_i^2/(2m)}, \qquad Z_x = \int \prod_{i=1}^{N} dx_i\, e^{-\beta \sum_i u(x_i - x_{i-1})}.$$

The momentum partition function Z_p: For each momentum degree of freedom,

$$\int_{-\infty}^{+\infty} e^{-\beta \frac{p_i^2}{2m}}\, dp_i = \sqrt{\frac{2\pi m}{\beta}}.$$

Therefore,

$$Z_p = \left(\sqrt{\frac{2\pi m}{\beta}}\right)^N.$$

The configurational partition function Z_x Introduce the bond variables

$$z_i = x_i - x_{i-1}, \qquad z_i > 0.$$

The Jacobian for the transformation $\{x_i\} \mapsto \{z_i\}$ is 1 (triangular transformation), hence

$$Z_x = \prod_{i=1}^{N} \int_0^\infty e^{-\beta\alpha z_i^4}\, dz_i = \left(\int_0^\infty e^{-\beta\alpha z^4}\, dz\right)^N.$$

Compute the integral with the substitution $u = \beta\alpha z^4$:

$$z = \left(\frac{u}{\beta\alpha}\right)^{1/4}, \qquad dz = \frac{1}{4}\left(\frac{1}{\beta\alpha}\right)^{1/4} u^{-3/4}\, du.$$

Thus

$$\int_0^\infty e^{-\beta\alpha z^4}\, dz = \frac{1}{4}\left(\frac{1}{\beta\alpha}\right)^{1/4} \int_0^\infty u^{-3/4} e^{-u}\, du = \frac{1}{4}\left(\frac{1}{\beta\alpha}\right)^{1/4} \Gamma\left(\frac{1}{4}\right),$$

and therefore

$$Z_x = \left[\frac{\Gamma(1/4)}{4}\left(\frac{1}{\beta\alpha}\right)^{1/4}\right]^N.$$

Helmholtz free energy: Using $Z = Z_p Z_x / h^N$,

$$\ln Z = N\left[\frac{1}{2}\ln\frac{2\pi m}{\beta} + \ln\frac{\Gamma(1/4)}{4} - \frac{1}{4}\ln(\beta\alpha) - \ln h\right].$$

The Helmholtz free energy

$$F = -\frac{1}{\beta}\ln Z = -N\frac{1}{4\beta}\left[\ln\frac{(2\pi m)^2}{\beta^3 \alpha h^4} + 4\ln\frac{\Gamma(1/4)}{4}\right].$$

3. The average energy is $\langle H\rangle = -\partial_\beta \ln Z$. Hence:

$$E = \langle H\rangle = N\left(\frac{1}{2\beta} + \frac{1}{4\beta}\right) = \frac{3N}{4\beta}.$$

4. The entropy is given by $S = -\partial F/\partial T$ or equivalently $S = (F - E)/T$. Hence:

$$S = -\frac{N\kappa_B}{4}\left[\ln\frac{(2\pi m)^2}{\beta^3\alpha h^4} + 4\ln\frac{\Gamma(1/4)}{4} - 3\right]$$

5. The average chain length is $L = \langle\sum_{i=1}^N (x_i - x_{i-1})\rangle = \sum_{i=1}^N \langle z_i\rangle$. Since the z_i are i.i.d. under the factorized measure, we have

$$L = N\langle z\rangle = N\frac{\int_0^\infty z\, e^{-\beta\alpha z^4}\, dz}{\int_0^\infty e^{-\beta\alpha z^4}\, dz} = N(\beta\alpha)^{-1/4}\frac{\Gamma(1/2)}{\Gamma(1/4)}$$

6. Since $u(z) = u(z-)$ we trivially have

$$L = N\langle z\rangle = N\frac{\int_{-\infty}^\infty z\, e^{-\beta\alpha z^4}\, dz}{\int_{-\infty}^\infty e^{-\beta\alpha z^4}\, dz} = 0.$$

   This is because the average position of each particle is fixed. On the other hand

$$L_2 = \sum_{i=1}^N \sqrt{\langle z_i^2\rangle} = N\sqrt{\langle z^2\rangle},$$

$$\langle z^2\rangle = \frac{\int_{-\infty}^\infty z^2\, e^{-\beta\alpha z^4}\, dz}{\int_{-\infty}^\infty e^{-\beta\alpha z^4}\, dz} = (\beta\alpha)^{-1/2}\frac{\Gamma(3/4)}{\Gamma(1/4)}.$$

   This leads to

$$L_2 = N(\beta\alpha)^{-1/4}\sqrt{\frac{\Gamma(3/4)}{\Gamma(1/4)}},$$

   which, a part from a numerical constant, agrees with the result of the previous point for the T-dependence.

5 Quantum Mechanics - Sextic quantum Hamiltonian

Giulio Ticli - University of Trieste

Consider the following one-dimensional quantum Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\hbar^2}{2m\lambda^2}\left(\frac{\hat{x}^6}{\lambda^6} - \frac{3\hat{x}^2}{\lambda^2}\right) \qquad \text{where } m > 0,\ \lambda > 0$$

1. Prove that $\hat{H}$ is positive-semi-definite. [7 points]
2. Prove that $\hat{H}$ has a non-degenerate spectrum in $L^2(\mathbb{R})$. [4 points]
3. Find the ground state $\psi_0(x)$, up to normalization. [7 points]
4. Normalize $\psi_0(x)$. [2 points]

Solution

Question a) For the sake of simplicity let $x = \lambda y$, so that:

$$\hat{H} = \frac{\hbar^2}{2m\lambda^2}\left(-\frac{\partial^2}{\partial y^2} + \hat{y}^6 - 3\hat{y}^2\right)$$

Define:

$$\hat{b} = \hat{y}^3 + \frac{\partial}{\partial y} \qquad \hat{b}^\dagger = \hat{y}^3 - \frac{\partial}{\partial y}$$

and notice that:

$$\hat{b}^\dagger\hat{b} = \hat{y}^6 + \left[\hat{y}^3, \frac{\partial}{\partial y}\right] - \frac{\partial^2}{\partial y^2} = \hat{y}^6 = \hat{y}^6 - 3\hat{y}^2 - \frac{\partial^2}{\partial y^2} \qquad \text{since} \qquad y^3\frac{\partial\psi}{\partial y} - \frac{\partial\left(y^3\psi\right)}{\partial y} = -3y^2\psi(y)$$

Therefore, for any ket $|\psi\rangle$:

$$\langle\psi|\,\hat{H}\,|\psi\rangle = \frac{\hbar^2}{2m\lambda^2}\langle\psi|\,\hat{b}^\dagger\hat{b}\,|\psi\rangle = \frac{\hbar^2}{2m\lambda^2}\left\|\hat{b}\psi\right\|^2 \geqslant 0$$

Question b) As all one-dimensional Hamiltonians of the form $\frac{\hat{p}^2}{2m} + V(\hat{x})$, this Hamiltonian has a non-degenerate spectrum in $L^2(\mathbb{R})$. Indeed, suppose that ψ and ζ are two eigenfunctions with the same energy ϵ. Then:

$$-\frac{\hbar^2}{2m}\psi''(x) + (V(x)-\epsilon)\,\psi(x) = 0 = -\frac{\hbar^2}{2m}\zeta''(x) + (V(x)-\epsilon)\,\zeta(x)$$

Multiplying the ψ side by ζ and the ζ side by ψ:

$$\begin{cases} -\frac{\hbar^2}{2m}\psi''(x)\zeta(x) + (V(x)-\epsilon)\,\psi(x)\zeta(x) = 0 \\ -\frac{\hbar^2}{2m}\psi(x)\zeta''(x) + (V(x)-\epsilon)\,\psi(x)\zeta(x) = 0 \end{cases}$$

Subtracting the two equations and dividing by some constants:

$$0 = \psi''(x)\zeta(x) - \psi(x)\zeta''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\psi'(x)\zeta(x) - \psi(x)\zeta'(x)\right) \implies \psi'(x)\zeta(x) - \psi(x)\zeta'(x) = k \text{ constant}$$

Since ψ and ζ are in $L^2(\mathbb{R})$, by taking the limit as $x \to \infty$ one sees that $k = 0$. Therefore:

$$0 = \frac{\psi'(x)\zeta(x) - \psi(x)\zeta'(x)}{[\zeta(x)]^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi}{\zeta}\right) \implies \psi \text{ is proportional to } \zeta$$

Question c) Since we know the spectrum is non-negative, non-degenerate and that $\hat{H}$ is proportional to $\hat{b}^\dagger\hat{b}$, if we can find a state $|\psi_0\rangle$ such that $\hat{b}\,|\psi_0\rangle = 0$, it must be the ground state. The differential equation:

$$y^3\psi_0(y) + \psi_0'(y) = 0$$

is easily solved:

$$\psi_0(y) = A\mathrm{e}^{-y^4/4} \qquad \text{that is} \qquad \psi_0(x) = A\mathrm{e}^{-x^4/4\lambda^4}$$

Question d) One must have:

$$\frac{1}{|A|^2} = \int_{-\infty}^{+\infty}\mathrm{e}^{-x^4/2\lambda^4}\,\mathrm{d}x = 2\int_0^{+\infty}\mathrm{e}^{-x^4/2\lambda^4}\,\mathrm{d}x = 2\int_0^{+\infty}\mathrm{e}^{-t}\frac{\lambda}{4}\sqrt[4]{\frac{2}{t^3}}\,\mathrm{d}t = 2^{-\frac{3}{4}}\lambda\int_0^{+\infty}\mathrm{e}^{-t}t^{\frac{1}{4}-1}\,\mathrm{d}t = 2^{-\frac{3}{4}}\Gamma\left(\frac{1}{4}\right)\lambda$$

Choosing without loss of generality $A \in \mathbb{R}^+$:

$$A = \frac{2^{3/8}}{\sqrt{\Gamma\left(\frac{1}{4}\right)\lambda}}$$

No points shall be awarded to those performing integration in $\mathrm{d}y$ without further multiplying by λ.